Reedy cats and Reedy structures: Kan's theorem and applications

References: • Hirschhorn "Model cats and their localizations"
• Emily Riehl "Categorical homotopy theory"

Lemma (Hirschhorn's lemma). A, B, X, Y are $\mathcal{D}$-diagrams in $\mathcal{M}$ (mod cat)

$\mathcal{D}$ is a Reedy cat

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

See Hirschhorn's Book, Chapter 15

Where h is defined on $F^{n-1}B$

$$\begin{tikzcd} A_\alpha \coprod_{L_\alpha A} L_\alpha B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B_\alpha \arrow[r] \arrow[ur, dashed, "H"] & Y_\alpha \times_{M_\alpha Y} M_\alpha X \end{tikzcd}$$

$\alpha \in \mathcal{D}$
$\deg(\alpha) = n$

There is a map $H: B_\alpha \to X_\alpha$ $\forall \alpha$ $\deg(\alpha) = n \iff h$ can be extended over the restriction of B to the n-filtration

Def. Let $\mathcal{D}$ be a Reedy cat, $\mathcal{M}$ is a modal cat

(1) $f: X \to Y$ is Reedy WE if $\forall \alpha \in \mathcal{D}_0$
$f_\alpha : X_\alpha \to Y_\alpha$ is a WE in $\mathcal{M}$

(2) $f: X \to Y$ is a Reedy cofibration if $\forall \alpha \in \mathcal{D}$
$$X_\alpha \coprod_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$$
is a cofibration in $\mathcal{M}$

(3) $f: X \to Y$ is a Reedy fibration if $\forall \alpha \in \mathcal{D}$

$$X_\alpha \longrightarrow Y_\alpha \times_{M_\alpha Y} M_\alpha X$$

is a fibration in $\mathcal{M}$

Theorem (D. Kan) (1) The cat $\mathcal{M}^{\mathcal{D}}$ with the Reedy WE, Reedy cofibs, Reedy fibs is a model cat

(2) If $\mathcal{M}$ is a left proper then the modal cat $\mathcal{M}^{\mathcal{D}}$ is also left proper model cat

Example. $\mathcal{M}^{\Delta^{op}}$

Lemma. (1) If $\forall \beta \in \mathcal{D}_0$ $\deg \beta < \deg \alpha$

$$X_\beta \sqcup_{L_\beta X} L_\beta Y \longrightarrow Y_\beta$$

has the LLP w.r. to S then

$$L_\alpha X \longrightarrow L_\alpha Y$$

has the LLP w.r. to S

(2) —〃—

Lemma. (1) If $\forall \alpha \in \mathcal{D}$

$$X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \to Y_\alpha$$

has the LLP vs. S $\Rightarrow$ $\forall \alpha$ the map $f_\alpha : X_\alpha \to Y_\alpha$ has the LLP w.r. to every element of S (pointwise)

Proof: $(f_\alpha : X_\alpha \to Y_\alpha) = (X_\alpha \to X_\alpha \sqcup_{\mathcal{L}_\alpha X} \mathcal{L}_\alpha Y \to Y_\alpha)$

$\Uparrow$ LLP as the pushout of $\mathcal{L}_\alpha X \to \mathcal{L}_\alpha Y$ $\qquad$ $\Uparrow$ LLP

$$\begin{tikzcd}
\mathcal{L}_\alpha X \arrow[r] \arrow[d] & \mathcal{L}_\alpha Y \arrow[d] \arrow[ddr, bend left] & \\
X_\alpha \arrow[r] \arrow[drr, bend right, "f_\alpha"'] & \cdot \arrow[ul, phantom, "\ulcorner"] \arrow[dr, "\exists!"] & \\
 & & Y_\alpha
\end{tikzcd}$$

$\triangleleft$

Prop. (1) If $f: X \to Y$ is a Reedy cofib $\Rightarrow$ f is a pointwise cofib and $\mathcal{L}_\alpha X \to \mathcal{L}_\alpha Y$ is cofib in $\mathcal{M}$ $\forall \alpha \in \mathcal{D}_0$

(2) $f: X \to Y$ is a Reedy fib $\Rightarrow$ f is a pointwise fib in $\mathcal{M}$ and $M_\alpha X \to M_\alpha Y$ is a fib in $\mathcal{M}$

Proof: f is a Reedy cofib $\Leftrightarrow$ $X_\alpha \sqcup_{\mathcal{L}_\alpha X} \mathcal{L}_\alpha Y \to Y_\alpha$ is a cofib in $\mathcal{M}$ $\Leftrightarrow$ the latter map has the LLP vs. the set of acyclic fibrations in $\mathcal{M}$ $\Leftrightarrow$ $f_\alpha : X_\alpha \to Y_\alpha$ has the LLP vs. the set of acyclic fibrations in $\mathcal{M}$ $\triangleleft$

Corollary (from the prop.) (1) If X is Reedy cofib $\Rightarrow$ $\forall \alpha \in \mathcal{D}_0$ both X_α and $\mathcal{L}_\alpha X$ are cofib in $\mathcal{M}$.

In particular, Reedy cofibrant diagrams are pointwise cofib

(2) —//—

Prop. (1) If $X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \to Y_\alpha$ is a trivial cofib $\Rightarrow \forall \alpha \in \mathcal{D}_0$ both f_α and $L_\alpha X \to L_\alpha Y$ are triv. cofibs

(2) —//— $\lhd$

$\rhd$ Obvious

Prop. (1) If $f: X \to Y$ cofib $\cap$ WE, $f_\alpha : X_\alpha \to Y_\alpha$, $L_\alpha f : L_\alpha X \to L_\alpha Y$, Then the maps

$X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \to Y_\alpha$ are trivial cofibs

(2) —//—

Proof: If $L_\alpha X \to L_\alpha Y$ is WE $\Rightarrow X_\alpha \sqcup_{L_\alpha X} L_\alpha X \to Y_\alpha$ will be a WE

$$\overbrace{\underbrace{X_\alpha \to X_\alpha \sqcup L_\alpha Y}_{WE} \to Y_\alpha}^{WE}$$

$X_\alpha \sqcup L_\alpha Y \to Y_\alpha$: WE by 2-of-3 property

- The base. If $\deg(\alpha) = 0 \Rightarrow L_\alpha X \to L_\alpha Y$ is an identity map
- The induction step. Suppose that $L_\beta X \to L_\beta Y$ is cofib $\cap$ WE $\forall \beta$, $\deg(\beta) < n$, let α be an object of

$\mathcal{D}$, $\deg(\alpha) = n$

$X_\beta \coprod_{L_\beta X} L_\beta Y \to Y_\beta$ is a trivial cofib $\forall \beta$, $\deg(\beta) < n$

$\Rightarrow L_\alpha X \longrightarrow L_\alpha Y$ is a triv. cofib. by Hirschhorn's Lemma $\triangleleft$

Theorem. (1) $f: X \to Y$ is in $\mathrm{Cofib} \cap \mathcal{WE}$

$\Leftrightarrow \forall \alpha \in \mathcal{D}_0$ $\quad X_\alpha \coprod_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$ is a trivial cofib in $\mathcal{M}$

(2) — " — (The Description of the trivial fibrations)

See Emily Riehl's Book, Chapter 14

Example. $0 \longleftarrow 1 \longrightarrow 2$

$$X : \underbrace{(0 \longleftarrow 1 \longrightarrow 2)}_{\mathcal{D}} \longrightarrow \mathcal{M}$$

$$b \xleftarrow{f} a \xrightarrow{g} c$$

is cofibrant in the Reedy model if a, b, c is cofib in $\mathcal{M}$ and g is cofib in $\mathcal{M}$

$X \in \mathcal{M}^{\mathcal{D}}$

$L_0 X = \emptyset$, $L_1 X = \emptyset$

$\overrightarrow{\mathcal{D}}_{<0}/0 \qquad \overrightarrow{\mathcal{D}}_{<1}/1$

$$L_2 X = \mathrm{colim}\left(\overrightarrow{\mathcal{D}}_{<2}/2 \xrightarrow{XU} \mathcal{M}\right) = a = X(1)$$

$X \to Y$ is a cofib in $\mathcal{M}^{\mathcal{D}} \iff \mathcal{L}_\alpha Y \sqcup_{\mathcal{L}_\alpha X} X_\alpha \to Y_\alpha$ is a cofib. in $\mathcal{M}$

Take X to be $\emptyset$

$$Y \text{ is cofib.} \iff \mathcal{L}_\alpha Y \sqcup_{\emptyset} \emptyset \to Y_\alpha$$

$$\emptyset = \mathcal{L}_\alpha \emptyset$$

$\mathcal{L}_0 X = \emptyset \longrightarrow b = X(0) \implies b$ should be cofibr.

$\mathcal{L}_1 X = \emptyset \longrightarrow a = X(1) \implies a$ —"—

$\mathcal{L}_2 X = a \longrightarrow c = X(2) \implies a \to c$ is a cofib.

Theorem. $- \otimes_{\mathcal{D}} - : sSet^{\mathcal{D}^{op}} \times \mathcal{M}^{\mathcal{D}} \longrightarrow \mathcal{M}$ is a left Quillen bifunctor w.r.t. the Reedy model cat

—"— $\{-,-\}^{\mathcal{D}} : (sSet^{\mathcal{D}})^{op} \times \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$

cotensor product

Corollary.

$$\begin{tikzcd}
& a \arrow[rr, tail, "g"] \arrow[dl, "f"'] \arrow[dd] & & c \arrow[dl] \arrow[dd] \\
b \arrow[rr, crossing over] \arrow[dd] & & \bullet & \\
& a' \arrow[rr] \arrow[dl] & & c' \arrow[dl] \\
b' \arrow[rr] & & \bullet \arrow[from=uu, dashed, crossing over] &
\end{tikzcd}$$

$\Rightarrow$ $* \otimes_{\mathcal{D}} \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ $\qquad *: \mathcal{D}^{op} \to sSet$

$\operatorname{colim}_{\mathcal{D}} : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ $\triangleleft$

Remark. In the left proper model cat setting the cofibrant objects condition can be dropped (Exercise)

What is a characterization of those cosimpl. sets that are Reedy cofibrant?

- Fix $X \in Set^{\Delta}$

$x \in X^n$ is non-degen. if it is not in the image of any monomorphism of Δ

$x = \sigma z$ where σ is a monomorphism in Δ

z is non-degen. simplex

- The nth latching map is a monomorphism when each such expr. is unique (See details in Riehl)

- if $x = \sigma z = \sigma' z \Rightarrow \sigma' = \sigma$

- $d^0, d^1 : [0] \rightrightarrows [1]$ — the exceptions from some rule

Def. If $\forall$ degen. simpl. in X is uniquely expr. as the image of a non-degen. simplex under a monomorphism we say that X has the Eilenberg-Zilber property

Prop. A cosimpl. object has the Eilenberg–Zilber prop.
$\iff$ it is unaugmentable, that is

$$eq\left(X^0 \rightrightarrows X^1\right) = \emptyset$$

Lemma. Any bisimpl. set is Reedy cofibrant

Lemma. If a cosimpl. object X is unaugmentable,
then $L_n X \to X$ is a monomorphism
If X and Y are both unaugmentable, then
any pointwise monomorphism $X \to Y$ is also a Reedy monic
monomorphism, i.e. its relative matching maps are monic

Sketch of Proof: A cosimpl. object in a Set-valued functor cat has the Eilenberg–Zilber property $\iff$ it does pointwise $\triangleleft$

Example. $L : \Delta \hookrightarrow sSet$, $L \in sSet^{\Delta}$
$eq\left(\Delta^0 \rightrightarrows \Delta^1\right) = \emptyset \implies L$ is cofibrant in the Reedy model structure in $sSet^{\Delta}$

Interesting question: What is the case when
$eq\left(X^0 \rightrightarrows X^1\right) \neq \emptyset$

Corollary. If M is a simplicial model cat
$|-| : M^{\Delta^{op}} \longrightarrow M$

is left Quillen with respect to the Reedy model structure

Proof. $|-| = - \otimes_{\Delta} \mathcal{L} \Rightarrow |-|$ is a left Quillen as $\mathcal{L}$ is cofibrant in the Reedy model structure and by the analog of Gambino's theorem $\triangleleft$

Homotopy limits in Top setting

- $|-| : \mathrm{Top}^{\Delta^{op}} \longrightarrow \mathrm{Top}$ preserves WE between so-called split simpl. spaces even are not not Reedy cofibrant
- $\mathcal{B}(*, \mathcal{D}, -) = |\mathcal{B}_{\bullet}(*, \mathcal{D}, -)|$

Def. A simpl. space $X_{\bullet}$ is split if $N_n X \hookrightarrow X_n$ $\forall n$, s.t.

$$\coprod_{[n] \twoheadrightarrow [k]} N_k X \hookrightarrow X_n$$

Example. $F : \mathcal{D} \to \mathrm{Top}$

$\mathcal{B}_{\bullet}(*, \mathcal{D}, F)$ is the split simpl. space

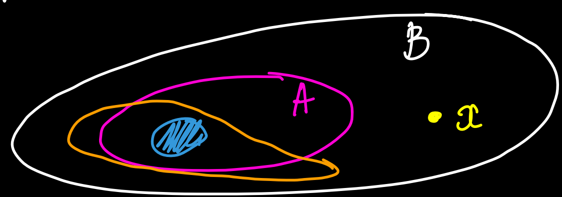
$$\mathcal{B}_{\bullet}(*, \mathcal{D}, F) = \coprod_{\vec{d} : [n] \to \mathcal{D}} F d_n$$

When $X_\cdot$ is split

$$\begin{tikzcd} |\partial\Delta^n| \times N_n X \arrow[r] \arrow[d] & sk_{n-1}|X| \arrow[d] \\ |\Delta^n| \times N_n X \arrow[r] & sk_n |X| \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Def. $A \hookrightarrow B$ is a relative T_1 inclusion

if $\forall\, U \subseteq A$, $\forall\, x \in B \setminus A$ $\exists\, V \subseteq B$



Lemma. If K is a compact

(Like small-object-argument-lemma)

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots$$

T_1-inclusions

$f : K \to \operatorname{colim}_n Y_n$ factors through some Y_k

Proof: Exercise

Prop. If $X_\cdot \to Y_\cdot$ is a pointwise WE of split simpl. spaces, then $|X_\cdot| \to |Y_\cdot|$ is a WE

Proof: • $X_\cdot$ is split $\Rightarrow$ $|X_\cdot|$ is a colimit of a sequence of relative T_1-inclusions

- By the lemma above

$$\operatorname*{colim}_n \pi_k |sk_n X_\cdot| \xrightarrow{\cong} \pi_k |X_\cdot|$$

$\forall k \geq 0$

- So, it suffice to show that $|sk_n X_\cdot| \to |sk_n Y_\cdot|$ are WE

$$\begin{tikzcd}
{|\Delta^n| \times N_n X} \arrow[d, "WE"'] & {|\partial\Delta^n| \times N_n X} \arrow[l] \arrow[r] \arrow[d, "WE"'] & {sk_{n-1}|X|} \arrow[d, "WE"'] \\
{|\Delta^n| \times N_n Y} & {|\partial\Delta^n| \times N_n Y} \arrow[l] \arrow[r] & {sk_{n-1}|Y|}
\end{tikzcd}$$

Lemma (WE have local property in Top)

A map $f : X \to Y$ in Top is WE if $\exists U, V$ - open cover of Y, s.t.

$$f^{-1}(U) \to U,\ f^{-1}(V) \to V,\ f^{-1}(U \cap V) \to U \cap V$$

are WE