# Motivation

- Models of $\infty$-categories
- Quasi-categories, complete Segal spaces, Segal categories, marked simplicial sets
- 2 category theory of $\infty$-categories
- Each of these forms a so-called $\infty$-cosmos
- This notion allows us to work with minimum of combinatorics
- An invariant or synthetic approach
- Functors between $\infty$-cosmos translate some important categorical properties such as adjunction, limits and colimits etc.

# Introducing quasi-categories

- One of the $\infty$-category model
- Quasi-categories, complete Segal spaces, Segal categories, marked simplicial sets
- **Def.** A quasi-category is simplicial set $X$ s.t.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad \forall\, n \geq 2,\quad 0 < k < n$$

inner horn inclusion

- **Examples** 1. Nerves of cats, each lift is unique
  2. Kan complexes (tautological)

# Homotopy category $hX$ of quasi-category

- **Def.** $Ob(hX) := X_0$

  $Mor(hX) := X_1/\sim$ homotopy relation

- $f \sim g \iff \exists$ 2-simplex with boundary having any of the forms:

$$\begin{tikzcd} & \bullet \arrow[dr, "f"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\sim"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, equal] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\sim"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\sim"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, equal] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\sim"] & & \bullet \end{tikzcd}$$

- Relations

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\sim"] & & \bullet \end{tikzcd}$$

## Closure properties

- One can form inner anodyne maps and inner fibrations
- They form the left and right classes of the weak factorization system
- These classes are closed under products, pullbacks, retracts and composition
- An important property:

If $X$ is quasi-cat and $A$ – simplicial set

$$\Downarrow$$

$X^A$ – quasi-cat

• Recall that if we have two-variable adjunction

$$-\otimes- : \mathcal{M}\times\mathcal{N}\longrightarrow\mathcal{P} \qquad \{-,-\} : \mathcal{M}^{op}\times\mathcal{P}\longrightarrow\mathcal{N}$$

$$\underline{hom}(-,-) : \mathcal{N}^{op}\times\mathcal{P}\longrightarrow\mathcal{M}$$

$$\mathcal{P}(m\otimes n, p)\cong\mathcal{N}(n,\{m,p\})\cong\mathcal{M}(m,\underline{hom}(n,p))$$

If $\mathcal{P}$ has pushouts and $\mathcal{M},\mathcal{N}$ have pullbacks then $\exists$ 2-var. adjunction

$$-\hat{\otimes}- : \mathcal{M}^2\times\mathcal{N}^2\longrightarrow\mathcal{P}^2 \qquad \widehat{\{-,-\}} : (\mathcal{M}^2)^{op}\times\mathcal{P}^2\longrightarrow\mathcal{N}^2$$

pushout-product

pullback-cotensor

$$\widehat{\underline{hom}}(-,-) : (\mathcal{N}^2)^{op}\times\mathcal{P}^2\longrightarrow\mathcal{M}^2$$

- For the pushout-product we have

$i : m \longrightarrow m' \in \mathcal{M}^2$

$j : n \longrightarrow n' \in \mathcal{N}^2$

$f : p \longrightarrow p' \in \mathcal{P}^2$

Pushout-product

$$
\begin{tikzcd}
m \otimes n \arrow[r, "i \otimes 1"] \arrow[d, "1 \otimes j"'] & m' \otimes n \arrow[d] \arrow[ddr, bend left, "1 \otimes j"] & \\
m \otimes n' \arrow[r] \arrow[drr, bend right, "i \otimes 1"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i \hat{\otimes} j"] & \\
 & & m' \otimes n'
\end{tikzcd}
$$

Pullback-cotensor

$$
\begin{tikzcd}
\{m', p\} \arrow[drr, bend left] \arrow[dr, dashed, "\widehat{\{i, f\}}"] \arrow[ddr, bend right] & & \\
 & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{m, p\} \arrow[d] \\
 & \{m, p\} \arrow[r] & \{m, p'\}
\end{tikzcd}
$$

Pullback-hom

$$
\begin{tikzcd}
hom(n', p) \arrow[drr, bend left] \arrow[dr, dashed, "\widehat{\underline{hom}}(j, f)"] \arrow[ddr, bend right] & & \\
 & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & hom(n, p) \arrow[d] \\
 & hom(n', p') \arrow[r] & \underline{hom}(n, p')
\end{tikzcd}
$$

- In particular, take sSet for $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & X^A \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \times \Lambda^n_k \arrow[r] \arrow[d] & X \\
A \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
 & X^{\Delta^n} \arrow[d] \\
A \arrow[r] \arrow[ur, dashed] & X^{\Lambda^n_k}
\end{tikzcd}
\quad \leftrightsquigarrow
$$

$$
\leftrightsquigarrow \quad
\begin{tikzcd}
\partial\Delta^m \arrow[rr] \arrow[d] & & X^{\Delta^m} \arrow[d] \\
\Delta^m \arrow[r] \arrow[urr, dashed] & A \arrow[r] & X^{\Lambda^n_k}
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\partial\Delta^m \times \Delta^n \coprod_{\partial\Delta^m \times \Lambda^n_k} \Delta^m \times \Lambda^n_k \arrow[r] \arrow[d, "i_m \hat{\times} j^n_k"'] & X \\
\Delta^m \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}
$$

- There are extensions above by the Joyal's result

**Prop. (Joyal)** The pushout-product of a monomorphism with an inner anodyne map is inner anodyne

**Sketch of proof**

- The bifunctor $-\hat{\times}-$ preserves colimits in each variable
- Due to the small object argument, decompose these monomorphisms into pushouts of inner horns
- So, it suffices to prove the statement for $i_m \hat{\times} j_k^n$ where

$$i_m : \partial\Delta^m \hookrightarrow \Delta^m$$
$$j_m : \Lambda^n_k \hookrightarrow \Delta^n$$

$\triangleleft$

- By two-variable adjunction one can also prove:
  - $\underline{\hom}(\underbrace{i}_{\text{cofib}}, \underbrace{f}_{\text{inner fibration}})$ is an inner fibration
  - $\underline{\hom}(\underbrace{j}_{\text{inner analyne}}, \underbrace{g}_{\text{inner fibration}})$ is a trivial fibration

**Corollary 1** If $A \in sSet$, $X \in qCat \Longrightarrow X^A \in qCat$

**Proof**

$$\begin{array}{c} X^A \\ \downarrow \\ * \end{array} = \underline{\hom}\left( \begin{array}{c} \emptyset \\ \downarrow \\ A \end{array}, \begin{array}{c} X \\ \downarrow \\ * \end{array} \right)$$

$$\begin{tikzcd}
\hom(A, X^A) \arrow[rr, bend left] \arrow[dr, dashed] \arrow[ddr, bend right] & & \hom(\emptyset, X^A) = \emptyset \arrow[dd] \\
& \bullet \arrow[ur] \arrow[d] & \\
& \hom(A, *) \arrow[r] & \hom(\emptyset, *) = \emptyset
\end{tikzcd}$$

$\triangleleft$

**Corollary 2** If $X \in qCat$, $\Lambda^n_k$ is inner horn $\Rightarrow X^{\Delta^n} \longrightarrow X^{\Lambda^n_k}$ is a trivial fibration

**Corollary 3** The fiber over any point is a contractible Kan complex, that is the space of fillers to a given horn in $X$ is a contractible Kan complex

"Well defined up to a contractible space of choises"

**Theorem** $\lim^W X \in qCat$ where $X : \underline{D} \longrightarrow \underline{qCat}$ – a diagram

– full simpl. subcat, spanned by quasi-categories

$W : \underline{D} \longrightarrow \underline{sSet}$ – a projectively cofib. weight

**Proof** • $\emptyset \to W$ is a retract of transfinite comp. of pushouts of coproducts of
$\underline{D}(d,-) \times \partial\Delta^n \longrightarrow \underline{D}(d,-) \times \Delta^n$, $n \geq 0$, $d \in \underline{D}$

• As $\lim^{\emptyset} X \longrightarrow \lim^{W} X = * \longrightarrow \lim^{W} X$

it suffices to prove that

$$\lim^{D(d,-)\cdot\Delta^n} X \longrightarrow \lim^{D(d,-)\cdot\partial\Delta^n} X$$

is an inner fibration

• But

$$\lim^{D(d,-)\cdot\Delta^n} X \cong \int_{e\in D} (Xe)^{D(d,e)\cdot\Delta^n} \cong \left(\int_{e\in D} (Xe)^{D(d,e)}\right)^{\Delta^n} \cong (Xd)^{\Delta^n}$$

products commute with ends

by ninja-Yoneda lemma

• So, we have a map

$$(Xd)^{\Delta^n} \longrightarrow (Xd)^{\partial\Delta^n}$$

$\partial\Delta^n \hookrightarrow \Delta^n$ — monomorphism

quasi-cats

• By Joyal's theorem it is an inner fibration

## Illustration for the previous theorem

$$f : 2 \to qCat, \quad \mathrm{Im}(f) = (f : X \to Y)$$

$$W = \underbrace{N(2/\_)}_{\text{proj. cofib}} : 2 \to sSet$$

$$\mathrm{Im}\big(N(2/-)\big) = (d^1 : \Delta^0 \to \Delta^1)$$

$$\begin{tikzcd}
\lim^{N(2/-)} f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\Delta^1} \arrow[d, "d^1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

- $\lim^{N(2/-)} f \cong Nf$ – the path space
- By the theorem $Nf$ is a quasi-cat

# Model structures

- Quillen model structure on sSet

  fibrant objects are Kan compexes

  cofibrations are monomorphisms

- Joyal model structure on sSet

**Theorem** The cofibrations and fibrant objects completely determine a model structure, supposing it exists

**Proof**

- We should find the weak equivalences
- Weak factorization system $(\mathcal{C}, \mathcal{F}_t)$ trivial fibrations

  cofibrations

- WFS with the fibr. obj. determine the weak equivalences

- WFS $\rightsquigarrow$ a cofibrant replacement notion
- By 2-of-3 a map is $\mathcal{WE} \iff$ cofib. rep. is $\mathcal{WE}$
- Hence, it suffices to determine the $\mathcal{WE}$ between cofib. objects
- Any model category $\mathcal{M}$ is saturated, that is

  a map $f$ in $\mathcal{M}$ is $\mathcal{WE} \iff f$ is an isomorphism in $h\mathcal{M}$
- Hence,

  $f : A \to B$ is $\mathcal{WE} \iff \operatorname{Ho}\mathcal{M}(B, X) \longrightarrow \operatorname{Ho}\mathcal{M}(A, X)$ – bijections for each fibrant $X$

  thanks to Yoneda lemma and

  $X \cong RX$

- So, we can suppose that $A$ and $B$ are cofibrant and apply Quillen's cylinder objects argument:

$$\mathrm{Ho}\,\mathcal{M}(A,X) = \mathcal{M}(A,X)/\sim$$

cofibrant

fibrant

the left homotopy relation by cylinder object for $A$

$\triangleleft$

**Notation** $\quad J := N(\mathbb{I}) = N\left(\bullet \underset{\cong}{\overset{\cong}{\rightleftarrows}} \bullet\right)$

free-standing isomorphism

- $J$ has only two non-degen. simplices in each dimension

"$J$ like $S^\infty$"

as

we have an action $\mathbb{Z}/2 \curvearrowright J$ permutes non-deg. simplicies

$$\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1) = B(*, \mathbb{Z}/2, *)$$

• By means of $\mathcal{I}$ we can form a cylinder object

"$\mathcal{I}$ like a segment"

• We have the cylinder objects (functorial)

$$A \sqcup A \rightarrowtail A \times \mathcal{I} \xrightarrow{\sim} A$$

**Lemma** The map $\mathcal{I} \to *$ is a trivial fibration

**Proof**

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{I} \arrow[d] \\ \Delta^n \arrow[r] & * \end{tikzcd}$$

• $n=0$ is OK since $\mathcal{I} \neq \emptyset$

• $\mathcal{I} = \mathrm{cosk}_0 \mathcal{I}$, i.e. $\mathcal{I}$ is 0-coskeletal as gruppoid

$$\begin{tikzcd} sk_0 \partial\Delta^n \arrow[r] \arrow[d, "\cong"'] & \mathcal{I} \arrow[d] \\ sk_0 \Delta^n \arrow[r] & * \end{tikzcd}$$

• Now use the adjunction $sk_0 \dashv \mathrm{cosk}_0$

$\triangleleft$

- Note that
$$\begin{tikzcd} A \times J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, "\sim"] \\ A \arrow[r] & * \end{tikzcd}$$
Hence, $A \times J \to A$ is a WE

- Also $A \sqcup A \to A \times J$ is mono $\Rightarrow$ it is a cofibration

qCat

- Consider the quotient $[A, X]$ by $f \sim g$ :

sSet

- Denote it by $[A, X]_J$

$$\begin{tikzcd} A \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ A \times J \arrow[r] & X \\ A \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} * \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ J \arrow[r] & X^A \\ * \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

- We have seen that

$$\underset{\text{sSet}}{f: A \to B} \text{ is WE} \iff [B,X]_J \longrightarrow [A,X]_J \text{ is a bijection } \forall X \in qCat$$

may be serve as definition of "categorical equivalence"

**Example** If $f: A \to B$ is inner fibration $\Rightarrow$ $f$ is categorical equivalence

$\triangleright$ - $\forall X \in qCat$ $\quad X^B \longrightarrow X^A$ is a triv. fib.

- $\Rightarrow$ it has a section $X^A \longrightarrow X^B$
- $\Rightarrow$ $[B,X]_J \longrightarrow [A,X]_J$ is surjective
- $\left(\begin{array}{c} * \sqcup * \\ \downarrow \\ J \end{array}\right) \boxslash \begin{array}{c} X^B \\ \downarrow \\ X^A \end{array} \Rightarrow [B,X]_J \longrightarrow [A,X]_J$ is injective $\triangleleft$

- In the same vein one can prove that

  the trivial fibrations are categorical equivalences

**Theorem (Joyal)** $\exists$ a left proper, cofib. gen., monoidal model structure on sSet:

fibrant objects are quasi-categories
cofibrations are monomorphisms
$\mathcal{WE}$ are categorical equivalences
fibrations between fibrant objects are isofibrations $\left( \Lambda^n_k \hookrightarrow \Delta^n \right) \boxslash -$ & $\left( * \to J \right) \boxslash -$

**Remark** There exists a set of generating trivial cofibrations, but no explicit description is known

# Quillen adjunction between models

**Theorem** homotopy category functor / nerve functor

$$h : sSet \rightleftarrows Cat : N$$ is a Quillen adjunction

Joyal's model structure / folk model structure

Recall: the folk model structure on Cat

WE — (usual) equivalence of categories

cofibrations — isofibrations $A \to B$

$A, B \in Cat$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

$$\mathbb{I} = \{\bullet \rightleftarrows \bullet\}$$

**Proof of theorem**

- $h$ sends monos to functors that are injective on objects
- It remains to prove that $N$ preserves fibrations
- $N(-)$ is fully faithful as $\varepsilon : hN \to id_{Cat}$ is an isomorphism
- $N(-)$ sends isofibrations in Cat to $(* \longrightarrow J)^{\boxslash}$

- Also $N(-)$ sends functors to an inner fibration
- So, by Joyal's result $N(-)$ preserves fibrations $\lhd$

**Corollary** 1) If $f: X \to Y$ is a categorical equivalence, then $hf: hX \longrightarrow hY$ is an equivalence of categories

2) And vice versa, if $F: \mathcal{C} \to \mathcal{D}$ is an equiv. of cats, then $NF: N\mathcal{C} \longrightarrow N\mathcal{D}$ is a categorical equivalence

**Proof** By the theorem above and Ken Brown's Lemma $\lhd$

# Model structures comparison

- Cofibs of Joyal's model structure $=$ Cofibs of Quillen's model structure
  Fibs of Joyal's model structure $\supset$ Fibs of Quillen's model structure

- Hence $W\mathcal{E}_J \subset W\mathcal{E}_Q$

- So, Quillen's model structure is a left Bousfield localization of Joyal's one

- As a consequence, WE between Kan complexes is a categorical equivalence. Moreover, it is an equivalence of quasi-categories

  it is not true in general : $\Delta^1 \to J \rightsquigarrow 2 \to \mathbb{I}$

# Mapping spaces

- We wanna form a hom-space between vertices $x, y \in \underset{\mathrm{qCat}}{\underset{\rotatebox[origin=c]{90}{$\in$}}{X}}$
- It would be cool if hom-space was a quasi-category!
- There is a quasi-category $X^{\Delta^1}$ whose $n$-simplices are

$$\Delta^n \times \Delta^1 \longrightarrow X$$

- Consider the pullback

$$\begin{tikzcd}
\mathrm{Hom}(x,y) \arrow[r] \arrow[d, swap, "\text{Fib as a pullback of Fib}"] \arrow[dr, phantom, very near start, "\lrcorner"] & X^{\Delta^1} \arrow[d, "\text{Fib in Joyal's model structure}"] \\
* \arrow[r, swap, "{(x,y)}"] & X \times X \cong X^{\partial \Delta^1}
\end{tikzcd}$$

- An $n$-simplex of $\mathrm{Hom}_X(x,y)$ is a map $\Delta^n \times \Delta^1 \to X$, s.t.

$\mathrm{Im}(\Delta^n \times \{0\})$ is degenerate at $x$

$\mathrm{Im}(\Delta^n \times \{1\})$ is degenerate at $y$

- 1-simplices:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] & y \arrow[d, equal] \arrow[dl, phantom, "\sim", shift left=3] \\ x \arrow[r, "g"'] \arrow[ur, phantom, "\sim", shift right=3] & y \end{tikzcd}$$

$\pi_0 \mathrm{Hom}_X(x,y) = \mathrm{Hom}_{hX}(x,y)$

- A more efficient construction: $\mathrm{Hom}^R_X(x,y)$

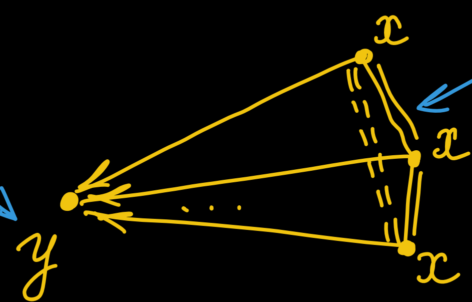
0-simplices are 1-simplices in $X$

1-simplices are 2-simplices of the form

$$\begin{tikzcd} & x \arrow[dr] & \\ x \arrow[ur, equal] \arrow[rr] \arrow[rr, phantom, "\sim", shift left=3] & & y \end{tikzcd}$$

$n$-simplices are $(n+1)$-simplices

the last vertex




the $(n+1)$th face

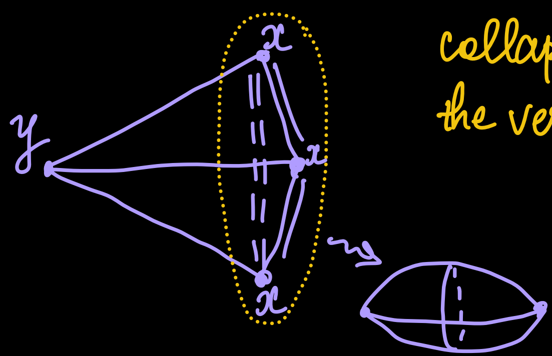
- $\mathrm{Hom}^L_X(x,y)$ is defined dually:

$$\mathrm{Hom}^L_X(x,y) = \left(\mathrm{Hom}^R_{X^{op}}(y,x)\right)^{op}$$

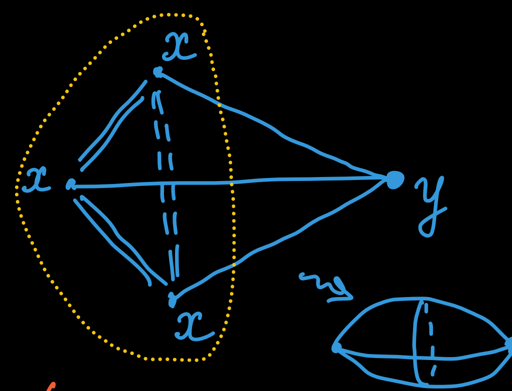
**Theorem** These models for the hom-space are categorically equivalent

- $n$-simplex in $\mathrm{Hom}^L_X(x,y)$ or in $\mathrm{Hom}^R_X(x,y)$ are given by the diagrams

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "d^0"'] & \Delta^0 \arrow[d] \\
\Delta^{n+1} \arrow[r] & \Delta^{n+1}_{0|1} =: C^n_L \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d] \\
\Delta^{n+1} \arrow[r] & \Delta^{n+1}_{n|n+1} =: C^n_R \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$



collapse the vertices



- The shape of $n$-simplex in $\mathrm{Hom}_X(x,y)$:

$$\begin{tikzcd}
\Delta^n \times \partial\Delta^1 \arrow[r, "\mathrm{proj}_2"] \arrow[d, "1\times i_1"'] & \partial\Delta^1 \cong * \sqcup * \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r] & C^n_{cyl} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

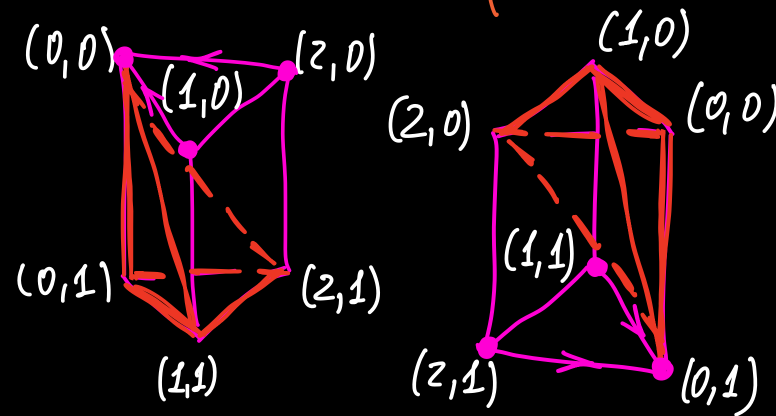
$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] \arrow[dr, phantom, "\sim", shift left=2ex] \arrow[dr, phantom, "\sim"', shift right=2ex] & y \arrow[d, equal] \\
x \arrow[r, "g"'] & y
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r] \arrow[r, bend right, "g"'] & \bullet
\end{tikzcd}$$

- We have canonical maps

$$\begin{tikzcd} C^n_{\mathcal{L}} \arrow[dr] & C^n_{cyl} \arrow[l, "r_{\mathcal{L}}"'] \arrow[r, "r_R"] \arrow[d] & C^n_R \arrow[dl] \\ & \Delta^1 & \end{tikzcd}$$

$r_{\mathcal{L}}, r_R : \Delta^n \times \Delta^1 \rightrightarrows \Delta^{n+1}$

— quotients of the unique retractions




- $C^\bullet_{\mathcal{L}}, C^\bullet_{cyl}, C^\bullet_R \in sSet^{\Delta}$

- The target of these cosimpl. objects is the category $\partial\Delta^1 / sSet =: sSet_{*,*}$

- $Ob(sSet_{*,*}) = \{$ quasi-category $X$ with chosen vertices $x, y \}$

- So,

$$Hom^{\mathcal{L}}_X(x,y) = sSet_{*,*}(C^\bullet_{\mathcal{L}}, X)$$

$$Hom_X(x,y) = sSet_{*,*}(C^\bullet_{cyl}, X)$$

$$Hom^R_X(x,y) = sSet_{*,*}(C^\bullet_R, X)$$

**Lemma** $C_R^\bullet$, $C_L^\bullet$ and $C_{Cyl}^\bullet$ are Reedy cofibrant

**Proof**

- Use the following fact:

  If a cosimplicial object $X$ is unaugmentable, then
  $$\mathcal{L}^n X \longrightarrow X$$
  is a monomorphism

- Recall that a cosimplicial object is an unaugmentable if
  $$eq\left( X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1 \right) = \text{the initial object}$$

- In our case for $C^\bullet_{Cyl}$
  $$C^0_{Cyl} \rightrightarrows C^1_{Cyl}$$
  include $C^0_{Cyl} = \Delta^1$ as the top and the bottom of
  $$\begin{tikzcd} \bullet \arrow[r] \arrow[d, equal] \arrow[dr, "\sim"] & \bullet \arrow[d, equal] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

- Hence, $eq(C^0_{Cyl} \rightrightarrows C^1_{Cyl}) = \partial\Delta^1$

  the initial object in $sSet_{*,*}$

$\triangleleft$

**Proposition** The canonical maps

$$C^{\bullet}_{\mathcal{L}} \longleftarrow C^{\bullet}_{cyl} \longrightarrow C^{\bullet}_{R}$$

are pointwise categorical equivalences

**Sketch of proof** From some combinatorial work one can obtain that the natural maps

$$C^{n}_{\mathcal{L}} \longrightarrow \Delta^1, \quad C^{n}_{cyl} \longrightarrow \Delta^1 \text{ and } C^{n}_{R} \longrightarrow \Delta^1$$

are categorical equivalences

$\triangleleft$

- Now prove the theorem
- Define for $A, X \in sSet_{*,*}$ their mapping space

$$\begin{tikzcd}
\underline{\hom}(A,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{A} \arrow[d, "X^{(a,b)}"] \\
* \arrow[r, "{(x,y)}"'] & X^{\partial\Delta^1}
\end{tikzcd}$$

$(a,b)$ and $(x,y)$ are base points

- If $X$ is a quasi-cat the functor

$$X^{(-)} : sSet^{op} \longrightarrow sSet$$

is right Quillen with respect to Joyal's model structure

- Given $A \to B$ in $sSet_{*,*}$

$$\begin{tikzcd}
 & \underline{hom}(A,X) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & X^A \arrow[dd] \\
\underline{hom}(B,X) \arrow[ur] \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & X^B \arrow[ur] \arrow[dd] & \\
 & * \arrow[rr] & & X^{\partial\Delta_1} \\
* \arrow[ur, equal] \arrow[rr] & & X^{\partial\Delta_1} \arrow[ur, equal] &
\end{tikzcd}$$

— a pullback square

- $\underline{hom}(-, X)$ is a pullback of $X^{(-)}$ $\Longrightarrow$ it defines a right Quillen functor

$$\underline{hom}(-, X) : sSet^{op}_{*,*} \longrightarrow sSet$$

- Now consider $C^\bullet : \Delta \longrightarrow sSet_{*,*}$

- $M_n \underline{hom}(C^\bullet; X) \cong \lim^{\partial\Delta^n} \underline{hom}(C^\bullet; X) \cong hom(\operatorname{colim}^{\partial\Delta^n} C^\bullet; X) \cong \underline{hom}(\mathcal{L}^n C^\bullet; X)$

- If $C^\bullet$ is Reedy cofibrant, the maps $\mathcal{L}^n C^\bullet \longrightarrow C^n$ are cofibrations

- Hence, the maps

$$\underline{hom}(C^n; X) \longrightarrow \underline{hom}(\mathcal{L}^n C^\bullet; X) \cong M_n \underline{hom}(C^\bullet; X)$$

are fibrations

- So, $\underline{hom}(C^\bullet; X)$ is Reedy fibrant with respect to the Joyal model structure

- We have pointwise equivalences between Reedy fibrant objects

$$\underline{hom}(C^\bullet_{\mathcal{L}}, X) \longrightarrow \underline{hom}(C^\bullet_{cyl}, X) \longleftarrow \underline{hom}(C^\bullet_{R}, X)$$

- But Reedy fibrant objects are pointwise fibrant

$\triangleleft$

- The rest of proof follows from the

**Lemma** $\exists f: X \to Y$ — WE between Reedy fibrant bisimplicial sets. Then the associated map of simpl. sets

$$X_{\bullet,0} \longrightarrow Y_{\bullet,0}$$

obtained by taking vertices pointwise is a WE

**Proof of lemma**

- By Ken Brown's Lemma it suffices to prove that if

$$f: X \longrightarrow Y$$

is a Reedy trivial fibration of Reedy fibr. bisimpl. sets then

$$X_{\bullet,0} \longrightarrow Y_{\bullet,0}$$

is an equivalence

- We will prove that $X_{\bullet,0} \to Y_{\bullet,0}$ is a trivial fibration
- $f$ is a Reedy trivial fibration $\iff$ $X_n \to Y_n \times_{M_n Y} M_n X$ is so in sSet
- It follows that

$$X_{n,0} \to (Y_n \times_{M_n Y} M_n X)_0 = Y_{n,0} \times_{(M_n Y)_0} (M_n X)_0$$ is a surjection in Set

- $(M_n X)_0 = \{\partial\Delta^n \to X_{\bullet,0}\}$ by the definition of matching object
- "Taking vertices pointwise" commutes with the weight limit as limits commute with limits
- By Yoneda lemma from the surjectivity we have a solution of a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X_{\bullet,0} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y_{\bullet,0} \end{tikzcd}$$

$\triangleleft$

Thank you!