Def. $E \in Sh(BornCoarse)$ coarsely invariant if $\forall X \in BornCoarse$

$$\{0,1\} \otimes X \longrightarrow X \rightsquigarrow E(X) \longrightarrow E(\{0,1\} \otimes X)$$

$$Sh^{\{0,1\}}(BornCoarse)$$

$$H^{\{0,1\}} : Sh(BornCoarse) \rightleftarrows Sh^{\{0,1\}}(BornCoarse) : 2$$

§4. Flasque Spaces

Def. $X \in BornCoarse$ is flasque if it admits $f : X \to X$ with

1. f & id_X are close to each other
2. $\forall U \in \mathcal{C}$ $\quad \bigcup_{k \in \mathbb{N}} (f^k \times f^k)(U) \in \mathcal{C}$
3. $\forall B \in \mathcal{B}$ $\exists k \in \mathbb{N}$ s.t.
$$f^k(X) \cap B = \emptyset$$

Flasqueness of X is implemented by f

Example. $[0,\infty)$ – a standard born. coarse space

X – a born. coarse space

$\Rightarrow \underbrace{[0,\infty)\otimes X}$ It is flasque

$f(t,x) := (t+1, x)$

- We want to move $[0,\infty)\otimes X$ to the right
- it is so as d on $[0,\infty)$ is transl. invariant

$[0,\infty)\times X$ is flasque since $f^k(x)\cap B \neq \emptyset$ $\forall k$

$B = [0,\infty)\times B_X$, $B_X \in \mathcal{B}_X$

$\mathbb{N}_d \subset [0,\infty)$

$\Rightarrow \mathbb{N}_d \otimes X$ is flasque $\forall X \in \mathrm{BornCoarse}$

Def. $E \in Sh^{\{0,1\}}(\mathrm{BornCoarse})$

E vanishes on a flasque born. coarse space

$E(X)$ is a final object in Spc $\forall X \in \mathrm{BornCoarse}$

Lemma. The coarsely invar. sheaves which vanishes on flasque spaces form a full localizing subcategory of $Sh^{\{0,1\}}(BornCoarse)$

$\triangleright$ E vanishes on $X \in BornCoarse$ – flasque

$\Leftrightarrow$ E is local w.r.t. $\mathcal{L}'(\emptyset) \longrightarrow \mathcal{L}'(X)$

$\mathcal{L}'(\emptyset) \cong \emptyset_{Sh}$ $\Rightarrow$ so, we should add such morphisms to our family of morphism $\triangleleft$

$$Fl: Sh^{\{0,1\}}(BornCoarse) \rightleftarrows Sh^{\{0,1\}, fl}(BornCoarse):incl$$

Example. If X – flasque $\Rightarrow$ $\emptyset \to X$ induces an equivalence

$$Fl(\mathcal{L}'(\emptyset)) \to Fl(\mathcal{L}'(X))$$

Def. (More general def. of flasqueness)

$X \in BornCoarse$

X – flasque in generalized sense if

$\exists (f_k)_{k \in \mathbb{N}}$ $\quad f_k : X \to X$ s.t.

1. $f_0 = id_X$
2. $\bigcup_{k \in \mathbb{N}} (f_k \times f_{k+1})(diag_X) \in \mathcal{C}_X$

3. $\forall\, U \in \mathcal{E}_X \quad \bigcup_{k \in \mathbb{N}} (f_k \times f_k)(U) \in \mathcal{E}_X$

4. $\forall\, B \in \mathcal{B}_X \quad \exists\, k_0 \in \mathbb{N}$ s.t. $\forall k \in \mathbb{N}\ k \geq k_0$

$$f_k(X) \cap B = \emptyset$$

Lemma. The following assertions are equiv.:

1. X is flasque in the generalized sense
2. $\exists\, F : \mathbb{N}_d \otimes X \to X$ s.t. $F \circ \iota = \mathrm{id}_X$

$$\iota : X \hookrightarrow \mathbb{N}_d \otimes X \text{ determined by } 0 \in \mathbb{N}$$

▷ $(1 \Rightarrow 2)$ X

$(f_k)_{k \in \mathbb{N}}$

$$F : \mathbb{N}_d \otimes X \to X \qquad F(k, x) := f_k(x)$$

- F - proper $\quad B \in \mathcal{B}_X \quad k_0 \in \mathbb{N} : \forall k \geq k_0 \ f_k(X) \cap B = \emptyset$

Then $F^{-1}(B) \subseteq \bigcup_{k=0}^{n-1} \{k\} \times f_k^{-1}(B)$ — bounded in $\mathbb{N}_d \otimes X$

- F - controlled

$$\mathcal{E}_{\mathbb{N}_d \otimes X} = \mathcal{E}\langle V \mid V := \{(n, n+1) \mid n \in \mathbb{N}\} \times \mathrm{diag}_X \rangle$$

$$\& \langle \mathrm{diag}_{\mathbb{N}} \times U \mid U \in \mathcal{E}_X \rangle$$

$$F(V) = \bigcup_{k\in\mathbb{N}} (f_k \times f_{k+1})(\mathrm{diag}_X) \in \mathcal{C}_X$$

by def.

$$F(\mathrm{diag}_{\mathbb{N}} \times U) = \bigcup_{k\in\mathbb{N}} (f_k \times f_k)(U) \in \mathcal{C}_X$$

$f_0 = id \Rightarrow F \circ 2 = id_X$

$1 \Leftarrow 2$) $F: \mathbb{N}_d \otimes X \to X$ s.t. $F \circ 2 = id_X$

$f_k(x) := F(k, x)$ $\triangleleft$

Prop. The property of being flasque in the generalized sense is coarsely invariant

$\triangleright$ X – is flasque in the gen. sense

$(f_k)_{k\in\mathbb{N}}$

$g: X \to Y$ – a coarse equiv.

$h: Y \to X$ – an inverse map

Define $(f'_k)_{k\in\mathbb{N}} : Y \to Y$ by $f'_0 = id_Y$

$f'_{k+1} = g \circ f_k \circ h$ – the desired family $\triangleleft$

Lemma. $E \in Sh^{\{0,1\}, fl}(BornCoarse)$

X — flasque in the generalized sense

$E(X) \simeq *$

▷ $(X \xrightarrow{id_X} X) = X \xrightarrow{2} \mathbb{N}_d \otimes X \xrightarrow{F} X$

$$\begin{tikzcd} E(X) \arrow[r] \arrow[rr, bend right, "id_{E(X)}"'] & \underbrace{E(\mathbb{N}_d \otimes X)}_{\cong\, *} \arrow[r] & E(X) \end{tikzcd}$$

So, $E(X) \cong *$ ◁

§5. u-Continuity & Motivic Coarse Spaces

$X \cong \operatorname{colim} X_U$

$X_U := (X, \mathcal{C}\langle U \rangle, \mathcal{B})$

We want to consider such functors $E \in Sh^{\{0,1\}, fl}(Born)$

Def.

$$E(X) \longrightarrow \lim_{U \in \mathcal{C}} E(X_U) \text{ — an equivalence}$$

$\forall (X, \mathcal{C}, \mathcal{B})$

We will say that E is u-continuous

Lemma. The full subcategory $\underbrace{Sp\mathcal{X}}_{\text{the category of motivic coarse spaces}}$ of $Sh^{\{0,1\},fl}(BornCoarse)$ of u-continuous sheaves is localizing

▷ Add the small set of morphisms

$$\operatorname*{colim}_{U \in \mathcal{C}} \mathcal{Y}'(X_U) \longrightarrow \mathcal{Y}'(X) \quad \forall X \in BornCoarse$$

to the list for which sheaves must be local ◁

$$U : Sh^{\{0,1\},fl}(BornCoarse) \rightleftarrows Sp\mathcal{X} : \text{incl.}$$

Def. $Y_0 := U \circ Fl \circ H^{\{0,1\}} \circ \mathcal{L} \circ \mathcal{Y}' : BornCoarse$

Remark. it can be omitted

$$\underset{BornCoarse}{\underset{\cap}{X}} \longmapsto \underset{Sp\mathcal{X}}{\underset{\cap}{Y_0(X)}}$$

Corollary: $Spc\,\mathcal{X}$ is presentable & fits into a localization:

1. $\mathcal{U}\circ Fl\circ H^{\{0,1\}}\circ \mathcal{L} : PSh(BornCoarse) \rightleftarrows Spc\,\mathcal{X} : incl.$

2. $(\mathcal{Z}, \mathcal{Y})$ – a compl. pair on $X \in BornCoarse$

$$\begin{tikzcd} Yo(\mathcal{Z}\cap\mathcal{Y}) \arrow[r] \arrow[d] & Yo(\mathcal{Y}) \arrow[d] \\ Yo(\mathcal{Z}) \arrow[r] & Yo(X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $Spc\,\mathcal{X}$

3. If $X \to X'$ is an equiv. $\Rightarrow$ $Yo(X) \to Yo(X')$ is an equivalence in $Spc\,\mathcal{X}$

4. If X – flasque $\Rightarrow$ $Yo(X)$ is an initial object in $Spc\,\mathcal{X}$

5. $\forall X$

$$Yo(X) \cong \operatorname*{colim}_{U\in\mathcal{C}} Yo(X_U)$$

Corollary: $Fun^{colim}(Spc\,\mathcal{X}, \mathcal{C}) \cong$ the full subcat of $Fun^{colim}(PSh(Born), \mathcal{C})$

$\mathcal{C}$ – an ∞-cat

What is the intuition of being presentable?

An ∞-cat is presentable iff it is equiv. to one of the form $\mathcal{P}(\mathcal{C}, R)$

- $\mathcal{C}$ — a small ∞-cat
- $R = \{f_i : X_i \to Y_i\}$ — a set of maps in

$$PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Groupoids)$$

- $\mathcal{P}(\mathcal{C}, R)$ — the full subcat of $PSh(\mathcal{C})$

spanned by F s.t.

$$Map(f, F) = PSh(Y, F) \longrightarrow PSh(X, F)$$

is an isomorphism of ∞-groupoids $\forall f \in R$

One can show that

$$\mathcal{P}(\mathcal{C}, R) \hookrightarrow PSh(\mathcal{C})$$

admits a left adjoint $\Rightarrow$ $\mathcal{P}(\mathcal{C}, R)$ is complete & cocomplete

$$Fun^{colim}(\mathcal{P}(\mathcal{C}, R), \mathcal{D}) \longrightarrow \widetilde{Fun}(\mathcal{C}, \mathcal{D})$$

it is a cocomplete ∞-cat

functors that send relations to isomorphisms

F ✓

$\hat{F}(f)$ is iso $\forall f$
$\quad \shortparallel$
$\mathrm{Lan}_z F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "z"'] & D \\ \mathrm{PSh}(\mathcal{C}) \arrow[ur, dashed, "\mathrm{Lan}_z F"'] & \end{tikzcd}$$

Motivic Coarse Spectra

Our aim: $\mathrm{Sp}\mathcal{X}$

$$\mathrm{Yo}^s : \mathrm{BornCoarse} \longrightarrow \mathrm{Sp}\mathcal{X}$$

§1. Stabilazation

$$\mathrm{Spc}^{la} \rightsquigarrow \mathrm{Sp}^{la}$$

$$\Sigma : \mathrm{Spc}^{la}_{*/} \longrightarrow \mathrm{Sp}^{la}$$

$$\begin{tikzcd} (* \to X) \arrow[r] \arrow[d] & (* \to *) \arrow[d, dashed] \\ (* \to *) \arrow[r, dashed] & \Sigma X \end{tikzcd}$$

$$\mathrm{Sp}^{la} := \mathrm{Spc}^{la}_{/*}[\Sigma^{-1}] := \mathrm{colim}\left(\mathrm{Spc}^{la}_{*/} \xrightarrow{\Sigma} \mathrm{Spc}_{*/} \xrightarrow{\Sigma} \cdots\right)$$

$$\Sigma^\infty_+ : \mathcal{S}pc^{la} \rightleftarrows \mathcal{S}p^{la} : \Omega^\infty$$

$$\Sigma^\infty_+ := (\mathcal{S}pc^{la} \to \mathcal{S}pc^{la}_{*/} \to \mathcal{S}p^{la})$$

Construct $\mathcal{S}p\mathcal{X}$

$$\Sigma^{mot}_+ := (\mathcal{S}pc\,\mathcal{X} \to \mathcal{S}pc\,\mathcal{X}_{*/} \xrightarrow{\Sigma} \mathcal{S}pc^{la})$$

$$\Sigma^{mot}_+ : \mathcal{S}pc\,\mathcal{X} \rightleftarrows \mathcal{S}p\mathcal{X} : \Omega^{mot}$$

<u>Def.</u> $\Upsilon^S_0(X) := \Sigma^{mot}_+(\Upsilon_0(X))$

§2. Homotopy invariance

$p_+ : X \to [0,\infty)$ $\quad$ $p_- : X \to (-\infty, 0)$

$\mathbb{R} \otimes X$

Def. The coarse cylinder $I_p X$

$$I_p X = \{ (t, x) \in \mathbb{R} \times X \mid p_-(x) \leq t \leq p_+(x) \} \subseteq \mathbb{R} \otimes X$$

Lemma. $I_p X \to X$ is a morphism

Prop. $Y_0^S(I_p X) \longrightarrow Y_0^S(X)$ is an equivalence

$I_p X$ – a cylinder over X

$$i_\pm : X \longrightarrow I_p X$$

$$i_\pm(x) = (p_\pm(x), x)$$

If $p_\pm$ are controlled

Def. $f_+ \,\&\, f_- : X \to X'$

They homotopic to each other if

$\exists\ p = (p_+, p_-) \quad h : I_p X \to X'$

$$f_\pm = h \circ i_\pm$$

$$\pi \circ i_\pm = id_X$$

$\Upsilon_0^S(i_+) = \Upsilon_0^S(i_-)$

Corollary: Suppose that f_+ & f_- are two homotopic maps $\Rightarrow$ $\Upsilon_0^S(f_+) \cong \Upsilon_0^S(f_-)$

Def. $f : X \longrightarrow X'$ is homotopy equiv.

$\exists g : X' \longrightarrow X$

$$f \circ g \simeq id_{X'} \;\&\; g \circ f \simeq id_X$$

Corollary: Υ_0^S sends equivalences to equivalences

Example. $n \in \mathbb{N}$, $n \geq 1$

$r : I \longrightarrow (0, \infty)$

$$X := \coprod_{i \in I} B^n(c_i, r(i))$$

$$\widetilde{U}_{r,i} = \bigcup_{i \in I} U_{r,i}$$

$U_{r,i}$ is an entourage U_r

$z : I \longrightarrow X$

$\pi : X \longrightarrow I$

$\pi \circ \imath = id_I$

$\imath \circ \pi = id_X$

$(i, x) \qquad x \in B(c_i, r(i))$

$p_- := 0 \quad p_+(i, x) := \|x\|$

One can form a coarse cylinder $I_p X$

$p = (p_-, p_+)$

$h : I_p X \longrightarrow X$

$$h(t, (i, x)) = \begin{cases} \left(i, x - t\frac{x}{\|x\|}\right), & x \neq 0 \\ (i, 0), & x = 0 \end{cases}$$

$\imath \circ \pi$ & id_X

$\Rightarrow Y_0^s(I) \cong Y_0^s(X)$

§3. Axioms for Coarse Homology Theory

$\mathcal{C}$ — cocomplete ∞-cat

$E: \mathrm{BornCoarse} \to \mathcal{C}$ — a functor

$\mathcal{Y} = (Y_i)_{i \in I}$

$$E(\mathcal{Y}) := \operatorname*{colim}_{i \in I} E(Y_i)$$

$$E(X, \mathcal{Y}) := \mathrm{Cofib}\big(E(\mathcal{Y}) \to E(X)\big)$$

Def. E is a coarse homology theory if

1. (excision) $\forall (\mathcal{Z}, \mathcal{Y})$

$$E(\mathcal{Z}, \mathcal{Z} \cap \mathcal{Y}) \longrightarrow E(X, \mathcal{Y})$$

is an equiv.

2. (coarse invariance) $X \to X'$ in BornCoarse

$E(X) \longrightarrow E(X')$ is an equiv. in $\mathcal{C}$

3. (vanishing on flasqueness)

If X — flasque $\Rightarrow E(X) \cong 0$

4. ($\mathcal{U}$-continuity)

$$X \cong \operatorname*{colim}_{U \in \mathcal{U}} X_U \rightsquigarrow E(X) \cong \operatorname*{colim}_{U \in \mathcal{U}} E(X_U)$$

The excision axiom can be replaced by

$$\begin{tikzcd}
E(Z \cap Y) \arrow[r] \arrow[d] & E(Z) \arrow[d] \\
E(Y) \arrow[r] & E(X)
\end{tikzcd}$$

is a pushout square