Representable Comma ∞-Categories

&

Model Independence

Representable Comma ∞-Categories

$$\begin{tikzcd} f\downarrow g \arrow[r] \arrow[d, "{(p_1, p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, "{(p_1, p_0)}"] \\ C\times B \arrow[r, "g\times f"'] & A\times A \end{tikzcd}$$

$f : B \to A$
$g : C \to A$ in ∞-cosmos $\mathcal{K}$

The top horizontal map corresponds to

$$\begin{tikzcd} & f\downarrow g \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[dr, "g"'] & \Leftarrow \varphi & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \qquad \mathrm{Hom}_A(f, g)$$

$$B = f\downarrow g \longrightarrow \begin{tikzcd} A^2 \arrow[r, bend left, "pr_0"] \arrow[r, bend right, "pr_1"'] \arrow[r, phantom, "\Downarrow\varphi"] & A \end{tikzcd}$$ The comma cone

$$\mathrm{Fun}(X, A^2) \xrightarrow{\cong} \mathrm{Fun}(X, A)^2$$

$$\begin{tikzcd} A^2 \arrow[r, bend left, "pr_0"] \arrow[r, bend right, "pr_1"'] \arrow[r, phantom, "\Downarrow\varphi"] & A \end{tikzcd}$$ – the canonical 2-cell

Def. Taking id morphisms, we will have

$$\begin{tikzcd} \mathrm{Hom}_A(f,A) := f\downarrow A \arrow[r] \arrow[d, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d] \\ A\times B \arrow[r, "\mathrm{id}_A\times f"'] & A\times A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_B(B,u) := B\downarrow u \arrow[r] \arrow[d, "{(q_1,q_0)}"'] & B^{\mathbb{2}} \arrow[d] \\ A\times B \arrow[r, "u\times \mathrm{id}_B"'] & B\times B \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Hom}_A(f,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ A \arrow[rr, phantom, "\Leftarrow\alpha", yshift=4ex] & & B \arrow[ll, "f"] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Hom}_B(B,u) \arrow[dl] \arrow[dr] & \\ B \arrow[rr, "u"'] \arrow[rr, phantom, "\Leftarrow\beta", yshift=4ex] & & A \end{tikzcd}$$

A left representation as a comma ∞-category

A right representation as a comma ∞-category

Recall the following weak universal properties:

- $$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, phantom, "\Leftarrow \chi" near start, shift left=3] & & B \arrow[ll, "f"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, "x"] & \\ A & f\downarrow A \arrow[l, "p_1"] \arrow[r, "p_0"'] & B \end{tikzcd} \Big/ \cong$$

fibered isomorphisms

$\chi : fb \Rightarrow a$

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A & \Leftarrow \chi & B \arrow[ll, "f"] \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d, dashed] \arrow[ddl, "a"'] \arrow[ddr, "b"] & \\ & f\downarrow A \arrow[dl, "p_1"'] \arrow[dr, Rightarrow, "p_0"] & \\ A \arrow[rr, phantom, "\Leftarrow \varphi", shift left=3] & & B \arrow[ll, "f"] \end{tikzcd}$$

In general case,

$$\left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] & \Leftarrow \alpha & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, "\ulcorner \alpha \urcorner"] & \\ C & \mathrm{Hom}_A(f,g) \arrow[l] \arrow[r] & B \end{tikzcd} \right\} \Big/ \cong$$

fibered isomorphismisms

$$\begin{tikzcd}
 & X \arrow[ddl] \arrow[ddr] \arrow[d, dashed, "\ulcorner \alpha \urcorner"] & \\
 & \mathrm{Hom}_A(f,g) \arrow[dl] \arrow[dr] & \\
C \arrow[dr] & \overset{\varphi}{\Leftarrow} & B \arrow[dl] \\
 & A &
\end{tikzcd}$$

Def. Given $C \xrightarrow{g} A \xleftarrow{f} B$

The comma ∞-category $\mathrm{Hom}_A(f,g) \longrightarrow C \times B$ is *left representable*

if $\exists\ \ell : B \longrightarrow C$ s.t.

$$\mathrm{Hom}_A(f,g) \underset{C\times B}{\simeq} \mathrm{Hom}_C(\ell, C)$$

& right representable if $\exists\ r : C \longrightarrow B$ s.t.

$$\mathrm{Hom}_A(f,g) \underset{C\times B}{\simeq} \mathrm{Hom}_B(B, r)$$

Our local goal: $\mathrm{Hom}_A(f,g)$ is right representable

$\Updownarrow$

$g: C \to A$ admits an absolute right lifting along $f: B \to A$

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

Three stages to do this:

① The 1st result characterizes those $\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow", near end] & A \end{tikzcd}$ that define absolute right lifting diagrams between comma ∞-categories as those that induce

$$\mathrm{Hom}_B(B, r) \underset{C \times B}{\cong} \mathrm{Hom}_A(f, g)$$

② The 2nd result: no natural transformation

$$\rho : fr \Rightarrow g$$

need be provided

③ The 3rd most general result: a criterion to construct a right representation to $\mathrm{Hom}_A(f, g)$ without a priori specifying functor r

Theorem

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A \end{tikzcd}$$ is an absolute right lifting diagram

$$\Updownarrow$$

$$\mathrm{Hom}_B(B,r) \underset{C\times B}{\simeq} \mathrm{Hom}_A(f,g)$$

$$\begin{tikzcd} & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow\varphi"] & \\ C \arrow[rr, "r"] \arrow[dr] & \arrow[d, phantom, "\Leftarrow\rho"] & B \arrow[dl] \\ & A & \end{tikzcd} \overset{(*)}{=} \begin{tikzcd} & \mathrm{Hom}_B(B,r) \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] \arrow[d, dashed, "y"] & \\ & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow\varphi"] & \\ C \arrow[dr] & {} & B \arrow[dl] \\ & A & \end{tikzcd}$$

Proof: ($\Rightarrow$) · Suppose that (r, ρ) defines an absolute right lifting of g through f

Recall: the univ. property of an ARL
$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r] \arrow[ur, phantom, shift right=2, "\Downarrow\lambda"] & A \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\swarrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow\zeta"] & B \arrow[d, "f"] \\ C \arrow[ur] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

- Apply it to the comma cone under $\mathrm{Hom}_A(f, g)$:

$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl] \arrow[dr] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\Leftarrow\varphi"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\exists!\Leftarrow\zeta"] & \\ C \arrow[dr, "g"'] \arrow[rr, "r"] \arrow[r, phantom, shift right=4, "\Leftarrow\rho"] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd} =$$

$$=$$

1-cell induction, applying to ς

$$
\begin{tikzcd}
 & \mathrm{Hom}_A(f,g) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_0"] \arrow[d, dashed, "Z"] & \\
 & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow \varphi"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & \Leftarrow \varsigma & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

$$=$$

substitute the equality (*) from the conditions of the theorem

$$
\begin{tikzcd}
 & \mathrm{Hom}_A(f,g) \arrow[d, dashed, "Z"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[dl] \arrow[dr] & \\
C \arrow[dr, "g"'] & \mathrm{Hom}_A(f,g) \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow \varphi"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

- Now, the functor $yZ : \mathrm{Hom}_A(f,g) \longrightarrow \mathrm{Hom}_A(f,g)$ factors the comma cone for $\mathrm{Hom}_A(f,g)$ through itself.
- So, $yZ \underset{C \times B}{\cong} id_{\mathrm{Hom}_A(f,g)}$

• It remains to show that $\mathcal{Z}y \underset{C\times B}{\cong} \mathrm{id}_{\mathrm{Hom}_B(B,r)}$

$$
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddddl, bend right] \arrow[ddddr, bend left] & \\
 & \mathrm{Hom}_A(f,g) \arrow[d, dashed, "\mathcal{Z}"] \arrow[dddl, bend right] \arrow[dddr, bend left] & \\
 & \mathrm{Hom}_B(B,r) \arrow[ddl] \arrow[ddr] & \\
 & \Leftarrow\varphi & \\
C \arrow[rr, "r"] \arrow[ddr, "g"'] & & B \arrow[ddl, "f"] \\
 & \Leftarrow\varrho & \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathrm{Hom}_A(f,g) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow\xi"] & \\
C \arrow[rr, "r"] \arrow[ddr, "g"'] & \, & B \arrow[ddl, "f"] \\
 & \Leftarrow\varrho & \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow\varphi"] & \\
C \arrow[rr] \arrow[ddr, "g"'] & \, & B \arrow[ddl, "f"] \\
 & \Leftarrow\varrho & \\
 & A &
\end{tikzcd}
$$

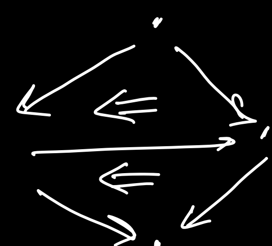
• It follows that $\varphi\mathcal{Z}y = \varphi$ from the univ. prop. of ARD & $\mathcal{Z}y = \mathrm{id}_{\mathrm{Hom}_B(B,r)}$

- So, we have shown that $Hom_B(B,r) \xrightarrow[\text{over } C\times B]{\simeq} Hom_A(f,g)$

($\Leftarrow$) Suppose that the functor $y : Hom_B(B,r) \longrightarrow Hom_A(f,g)$ is a fibered equivalence over $C\times B$

Prove that (r,ρ) is an absolute right lifting of g through f

- By uniqueness property of comma ∞-categories, the natural transf.



inherits the weak univ. prop. of a comma cone from $Hom_A(f,g)$

Also, we have

$$\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\Leftarrow\alpha"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}\right\}
\longleftrightarrow
\left\{\begin{tikzcd}
& X \arrow[d, dashed, "a"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\
& \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow\varphi"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A \arrow[u, phantom, "\Leftarrow\rho"] &
\end{tikzcd}\right\}\Big/\cong$$

$$\begin{tikzcd}
& \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow\varphi"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A \arrow[u, phantom, "\Leftarrow\rho"] &
\end{tikzcd}
= \rho p_1 \cdot f\varphi : fp_0 \Rightarrow gp_1$$

$$fp_0 \rightsquigarrow frp_1 \rightsquigarrow gp_1$$

$$\left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[rr, "r"] & & B \\ & \mathrm{Hom}_B(B,r) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] \arrow[u, phantom, "\Leftarrow \varphi"] & \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, phantom, "\Leftarrow \xi"] & \\ C \arrow[rr, "r"'] & {} & B \end{tikzcd} \right\}$$

φ is a right comma cone

- $\forall$ 2-cell on the right side produces a 2-cell on the left by pasting with ρ $\Longleftrightarrow$ we have the univ. prop. of absolute right lifting diagrams

Corollary The condition functor $f: A \longrightarrow B$ between ∞-cats being fully faithful is equivalent to each of the items:

(i)
$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] \arrow[ur, phantom, shift left=3, "\Vert"] & B \end{tikzcd}$$

The id defines an absolute right lifting diagram

(ii)
$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] \arrow[ur, phantom, shift left=3, "\Vert"] & B \end{tikzcd}$$

—— 〃 —— 〃 —— left lifting diagram

(iii) $\forall$ ∞-cat X the induced functor

$$f_*: h\mathrm{Fun}(X, A) \longrightarrow h\mathrm{Fun}(X, B)$$

is a fully faithful functor of 1-cats

(iv)

$$\begin{tikzcd}
 & A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, "2\,\ulcorner f\kappa \urcorner"] & \\
A & & A \\
 & \mathrm{Hom}_B(f,f) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] &
\end{tikzcd}$$

it is induced by the identity 2-cell id_f

$$\Updownarrow$$

$$\begin{tikzcd}
 & A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
A \arrow[dr, "f"'] \arrow[rr, phantom, "= \; id_f"] & & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

Proof: • Obviously, (i) & (ii) are equivalent with (iii)

• (i) & (ii) are equivalent with (iv) by the previous theorem $\triangleleft$

Corollary Cosmological functors preserve absolute lifting diagrams

Proof: • Let $F : \mathcal{K} \to \mathcal{L}$ be a cosm. fun. together with

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

• It induces a fibered equivalence

$$y : Hom_B(B, r) \xrightarrow[C \times B]{\sim} Hom_A(f, g)$$

• Applying F, we will have

since F preserves WE

$$Fy : Hom_{FB}(FB, Fr) \xrightarrow[FC \times FB]{\cong} Hom_{FA}(Ff, Fg)$$

$$
\begin{tikzcd}
 & \mathrm{Hom}_{FB}(FB, Fr) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow F\varphi"] & \\
FC \arrow[rr, "Fr"] \arrow[dr, "Fg"'] & \arrow[d, phantom, "\Leftarrow F\rho"] & FB \arrow[dl, "Ff"] \\
 & FA &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathrm{Hom}_{FB}(FB, Fr) \arrow[d, "\wr\ Fr"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathrm{Hom}_{FA}(Ff, Fg) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
FC \arrow[dr, "Fg"'] & \Leftarrow F\varphi & FB \arrow[dl, "Ff"] \\
 & FA &
\end{tikzcd}
$$

in $\mathcal{L}$

By the theorem this fibered equivalence witnesses the fact that

$$
\begin{tikzcd}
 & FB \arrow[d, "Ff"] \\
FC \arrow[ur, "Fr"] \arrow[r, "Fg"'] \arrow[ur, phantom, shift right=3, "\Downarrow F\rho"] & FA
\end{tikzcd}
$$

defines an absolute right lifting diagram in $\mathcal{L}$

$\lhd$

Theorem Given $r : C \to B$, $f : B \to A$ & $g : C \to A$

$$\left\{\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd}\right\} \longleftrightarrow \left\{\begin{tikzcd} & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, "y"] & \\ C & & B \\ & \mathrm{Hom}_A(f,g) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}\right\} \Big/ \cong$$

$$\begin{tikzcd} & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[rr, "r"] \arrow[rr, phantom, shift left=4, "\overset{\varphi}{\Leftarrow}"] \arrow[rr, phantom, shift right=4, "\overset{\rho}{\Leftarrow}"] \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & \mathrm{Hom}_B(B,r) \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] \arrow[d, dashed, "y"] & \\ & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\varphi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

Moreover, ρ is an abs. right lift of g through f $\Longleftrightarrow$ y is an equivalence (we have proved it)

Corollary Given $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$, there are bijections

$$
\left\{
\begin{tikzcd}
 & \mathrm{Hom}_B(g,B) \arrow[dl] \arrow[dr, two heads] \arrow[dd, "\ulcorner\alpha^*\urcorner"] & \\
B & & A \\
 & \mathrm{Hom}_B(f,B) \arrow[ul] \arrow[ur, two heads] &
\end{tikzcd}
\right\}_{/\cong}
\longleftrightarrow
\left\{
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
\right\}
\longleftrightarrow
\left\{
\begin{tikzcd}
 & \mathrm{Hom}_B(B,f) \arrow[dl] \arrow[dr, two heads] \arrow[dd, "\ulcorner\alpha_*\urcorner"] & \\
A & & B \\
 & \mathrm{Hom}_B(B,g) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\right\}
$$

They are constructed by pasting with the left/right comma cone over g/f resp.

$$
\begin{tikzcd}
 & \mathrm{Hom}_B(g,B) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\Leftarrow\varphi"] & \\
B & & A \arrow[ll, "g"'] \arrow[ll, bend left=40, "f"] \arrow[ll, phantom, bend left=20, "\alpha\Uparrow"]
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(g,B) \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[d, dashed, "\ulcorner\alpha^*\urcorner"] & \\
 & \mathrm{Hom}_B(f,B) \arrow[dl] \arrow[dr] & \\
B & & A \arrow[ll, "f"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{Hom}_B(B,f) \arrow[dl] \arrow[dr, two heads] \arrow[d, phantom, "\Leftarrow\varphi"] & \\
A \arrow[rr, "f"] \arrow[rr, bend right=40, "g"'] \arrow[rr, phantom, bend right=20, "\Downarrow\alpha"] & & B
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,f) \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[d, dashed, "\ulcorner\alpha_*\urcorner"] & \\
 & \mathrm{Hom}_B(B,g) \arrow[dl] \arrow[dr] & \\
A \arrow[rr] & & B
\end{tikzcd}
$$

- The next theorem allows us to recognize when a comma ∞-cat is right representable <u>in the absence of a predetermind representing functor</u>

- It spcializes to give existence theorems for adjoint functors & for (co)limits

<u>Theorem</u> $\mathrm{Hom}_A(f,g)$ associated to a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ is right representable $\iff \exists i$ — a <u>right adj. right inverse</u>

$\exists$ left adj. & ε is invertible

The nat. transf. is encoded by the functor $i : C \to \mathrm{Hom}_A(f,g)$

$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[ur, dashed, bend right, "i"'] & & B \end{tikzcd}$$

$p_1 i \cong id_C$

$p_0 i : C \to B$ defines the representing functor

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "p_0 i"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & A \end{tikzcd}$$

Example If $\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd}$ then

$$\begin{tikzcd} f\downarrow A \arrow[rr, "\cong"] \arrow[dr, "(p_1, p_0)"'] & & B\downarrow u \arrow[dl, "(q_1, q_0)"] \\ & A \times B & \end{tikzcd}$$

Proof: We know that

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\varepsilon"] & A \end{tikzcd}$$

are an absolute right lifting properties

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd}$$ is an absolute right lifting diagram

$$\Updownarrow$$

$$\mathrm{Hom}_B(B, r) \underset{C \times B}{\cong} \mathrm{Hom}_A(f, g)$$

Apply the first theorem

$$\mathrm{Hom}_B(B, u) \cong \mathrm{Hom}_A(f, \mathrm{id}_A) = \mathrm{Hom}_A(f, A)$$ $\triangleleft$

Example If
$$\begin{tikzcd} B \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\perp"] & & A \arrow[ll, bend left, "u"] \end{tikzcd}$$
then $\forall a : 1 \to A$
$\forall b : 1 \to B$

$$fa \downarrow b \cong a \downarrow ub$$

Proof:

- Fibered equivalences can be pulled back
- So, pull back along $1 \xrightarrow{(a,b)} A \times B$:

$$\begin{tikzcd}
fb\downarrow a \arrow[rr] \arrow[dr, "\cong"] \arrow[ddr, bend right] & & f\downarrow A \arrow[dr, "\cong"] \arrow[ddr, bend right] & \\
& a\downarrow ub \arrow[rr] \arrow[d] & & B\downarrow u \arrow[d] \\
& 1 \arrow[rr, "{(a,b)}"] & & A \times B
\end{tikzcd}$$

$\triangleleft$

Adjunctions, (co)limits via commas

- We know that adjunctions can be encoded via isomorphisms of commas
- Also, it is true for (co)limits:

Def (the ∞-cat of cones)

Given an J-indexed diagram $d: 1 \longrightarrow A^J$ in an ∞-cat A, the ∞-cat of cones over d:

$$\begin{tikzcd} & \mathrm{Hom}_{A^J}(\Delta, d) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ 1 \arrow[dr, "d"'] & \arrow[l, phantom, "\overset{\varphi}{\Leftarrow}"] & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd}$$

$\Delta : A \longrightarrow A^J$ is constructed by applying the bifunctor $(J, A) \mapsto A^J$ to $! : J \to 1$

From the first main theorem we get

Prop. $\ell : 1 \longrightarrow A$ defines a limit for a diagram $d : 1 \longrightarrow A^{\mathcal{J}} \iff \exists$ a fibered equivalence

$$\begin{tikzcd}
\mathrm{Hom}_{A^{\mathcal{J}}}(A, \ell) \arrow[rr, "\cong"] \arrow[dr, "p_1"'] & & \mathrm{Hom}_{A^{\mathcal{J}}}(\Delta, d) \arrow[dl, "p_0"] \\
& A &
\end{tikzcd}$$

Proof: Recall by def:

$\lim d$ is the limit of a diagram $d : 1 \to A^{\mathcal{J}}$

$$\Updownarrow$$

$$\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\varepsilon", shift right=3] & A^{\mathcal{J}}
\end{tikzcd}$$ is an absolute right lifting diagram $\triangleleft$

Prop. ((co)limits represent cones) A family of diagrams $d : D \to A^{\mathcal{J}}$ admits a limit $\iff$ $\mathrm{Hom}_{A^{\mathcal{J}}}(\Delta, d)$ is right presentable

$$\mathrm{Hom}_{A^{\mathcal{J}}}(\Delta, d) \underset{D \times A}{\cong} \mathrm{Hom}_A(A, \ell)$$

$\ell : D \longrightarrow A$ defines the limit functor

Dually, $\exists \operatorname{colim}(d : D \longrightarrow A^{\mathcal{J}}) \iff \mathrm{Hom}_{A^{\mathcal{J}}}(d, \Delta)$ is left repr.

$$\mathrm{Hom}_{A^{\mathcal{J}}}(d, \Delta) \underset{A \times D}{\cong} \mathrm{Hom}_A(c, A)$$

$c : D \longrightarrow A$ defines the colimit functor

Prop. (limits are terminal cones) A diagram $d: 1 \to A^J$ in an ∞-cat A

(i) admits a lim $\iff$ $\mathrm{Hom}_{A^J}(\Delta, d)$ admits a terminal element

(ii) admits a colim $\iff$ $\mathrm{Hom}_{A^J}(d, \Delta)$ admits an initial element

Model independence of basic ∞-cat theory

- Any categorical property that can be captured by the existence of a fibered equivalence between comma ∞-cats is "model independent"

preserved by any cosmological functor

&

reflected by those that define WE of ∞-cosmoi

Def (Recall) A cosmological functor is a simplicial functor between ∞-cosmoi that preserves the class of isofibrations & terminal object 1, cotensors A^U

U ← sSet

A ← an object of ∞-cosmos $\mathcal{K}$

- Also, it preserves $\mathcal{WE}$ & $\mathcal{F}ib^{tr}$
- A cosmological functor $F : \mathcal{K} \longrightarrow \mathcal{L}$ induces a 2-functor

$$F_2 := h_{o*}F : \mathcal{K}_2 \longrightarrow \mathcal{L}_2$$

$$\mathcal{K}_2 = h_{o*}\mathcal{K}, \qquad \mathcal{L}_2 = h_{o*}\mathcal{L}$$

Prop. $F: \mathcal{K} \longrightarrow \mathcal{L}$ induces $F_2 : \mathcal{K}_2 \longrightarrow \mathcal{L}_2$ that preserves adjunctions, equivalences, isofibrations, trivial fibrations, groupoidal objects, products & comma objects

Proof:

- Any 2-functor preserves adj. & equiv.
- Isofibrations are preserved by the def. of cosm. functors
- $\mathrm{Fib}^{tr} = \mathrm{Isofib} \cap \mathrm{Equiv.} \Longrightarrow \mathrm{Fib}^{tr}$ are preserved as well
- E is groupoidal $\Longleftrightarrow$ $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$ is in Fib^{tr}

⌜Recall: groupoidal objects in an ∞-cosmos $\mathcal{K}$

(i) E is groupoidal

(ii) $\forall$ 2-cell with codomain E is inv. in $\mathcal{K}_2$

(iii) $\forall X \in \mathcal{K}$ $\mathrm{Fun}(X, E)$ is a Kan complex

(iv) $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$ is in Fib^{tr} ⌟

⌜ (i) $\Rightarrow$ (ii) obviously by def.

(ii) $\Leftrightarrow$ (iii) by Joyal

(iv) $\Leftrightarrow$ $\mathrm{Fun}(X,E)^{\mathbb{I}} \twoheadrightarrow \mathrm{Fun}(X,E)^{2}$ is in $\mathrm{Fib}^{\mathrm{tr}}$ $\forall X$

Surj. on vertices $\Rightarrow$ $\forall$ 1-simplex in $\mathrm{Fun}(X,E)$ is an iso

And we have (iii)

(iii) $\Rightarrow$ (iv) $\;2 \hookrightarrow \mathbb{I}$ is a weak homotopy equiv. ⌟

- Preservation of commas — by def. of commas

  &

  by uniqueness of commas

◁

Def (Weak equivalences of ∞-cosmoi)

$F : \mathcal{K} \to \mathcal{L}$ is WE when it is

(a) surjective on objects up to equivalence

$$\forall X \in \mathcal{L} \quad \exists A \in \mathcal{K} \text{ s.t. } FA \simeq X \in \mathcal{L}$$

(b) a local equivalence of quasi-categories:

$$\mathrm{Fun}(A, B) \xrightarrow{\simeq} \mathrm{Fun}(FA, FB) \text{ — an equiv. of quasi-cats}$$

$\forall A, B \in \mathcal{K}$

Prop. If F is a WE of ∞-cosmoi then F_2

(i) defines a biequivalence $F_2 : \mathcal{K}_2 \to \mathcal{L}_2$

F_2 is surj. on obj. & $\hom(A,B) \xrightarrow{\simeq} \hom(FA,FB)$ $\forall A, B \in \mathcal{K}$

(ii) $\hom(A,B) \xrightarrow{\cong} \hom(FA,FB)$ induces a bijection on isomorphism classes of objects

(iii) preserves & reflects groupoidal objects:

$A \in \mathcal{K}$ is groupoidal $\iff$ $FA \in \mathcal{L}$ is so

(iv) preserves & reflects equivalences

$A \simeq B \in \mathcal{K} \iff FA \simeq FB \in \mathcal{L}$

(v) preserves & reflects comma objects:

given $E \to C \times B$ & $C \xrightarrow{g} A \xleftarrow{f} B$ in $\mathcal{K}$

then

$$E \underset{C\times B}{\cong} \mathrm{Hom}_A(f,g) \iff FE \cong \mathrm{Hom}_{FA}(Ff, Fg)$$
$$\underset{FC\times FB}{\cong} F\left(\mathrm{Hom}_A(f,g)\right)$$

Proof: • (i) – (iii) are obvious

• The preservation halves of (iv) – (vi) was proved in the previous prop.

• The reflection part of (iv) – (vi) is obvious $\triangleleft$

Theorem (model independence of basic category theory I)

The following notions are preserved & reflected by any WE of ∞-cosmoi:

(i) The adjointness
$$\begin{tikzcd} B \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\perp"] & & A \arrow[ll, bend left, "u"] \end{tikzcd}$$

(ii) The existence of left & right adj. to $u: A \to B$

(iii) The question of whether a given element $\ell: 1 \to A$ defines a limit or a colimit for a diagram $d: 1 \to A^J$

(iv) The existence of a limit or a colimit for a J-indexed diagram $d: 1 \longrightarrow A^J$ in an ∞-cat A

Proof: These notions can be expressed via commas $\triangleleft$

Thank you!