Simplicial Categories & Homotopy Coherence

How to produce quasi-categories?

- The answer is: by means of the adjunction

$$\mathfrak{C} \dashv \mathfrak{N}$$

the functor producing simplicial categories

homotopy coherent nerve

- The aime of this talk is to introduce such functors

Recall: simplicial functors

- A simplicial functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ consists of functors $F_n : \mathcal{C}_n \to \mathcal{D}_n$ for each n that commute with the simplicial operator functors

- Example Consider a simplicial functor

$$F : \underline{\mathcal{C}} \longrightarrow \underline{sSet}$$

for each $x \in \underline{\mathcal{C}}$ we specify Fx together with a map $\Delta^n \times Fx \to Fy$ $\forall n$-simplex in $\underline{\mathcal{C}}(x,y)$, s.t. the faces and degen. of $\Delta^n \times Fx \to Fy$ correspond to the ones of the n-simpl.

Recall: simplicial natural transformations

- $F, G: \underline{\mathcal{C}} \Longrightarrow \underline{\mathcal{D}}$

is given by arrows in the underlying category of $\underline{\mathcal{D}}$ for each object of $\underline{\mathcal{C}}$, s.t. $\alpha_x \in \mathcal{D}_0(Fx, Gx)$ $\forall x \in \mathcal{C}$:

- α_x form a natural transformations between F_0 & G_0
- $s_0(\alpha_x) \in \mathcal{D}_1(Fx, Gx)$ should form $F_1 \Rightarrow G_1$
- $s_0 s_0(\alpha_x) = s_1 s_0(\alpha_x) \in \mathcal{D}_2(Fx, Gx) \rightsquigarrow F_2 \Rightarrow G_2$
- the images of the α_x under the unique degeneracy operator $[n] \twoheadrightarrow [0]$ form a natural transformation $F_n \Rightarrow G_n$

Some natural functors

- Consider functors

$c : Set \longrightarrow sSet$ — constant functor

$\pi_0 : sSet \longrightarrow Set$ — homotopy functor

$ev_0 : sSet \longrightarrow Set$ — underlying category functor

$\ddot{X} \qquad \Delta^0 \longrightarrow X$

- These functors are monoidal
- So, they induced the functors c_*, $(\pi_0)_*$ and $(ev_0)_*$ between enriched categories

Some natural functors

- $\mathfrak{C} = \mathfrak{C}_* : \mathcal{C}at \longrightarrow \mathcal{C}at_\Delta$

  cat of small cats — cat of simplicial cats

- $\pi = (\pi_0)_* : \mathcal{C}at_\Delta \longrightarrow \mathcal{C}at$

- $\mathcal{U} = (ev_0)_* : \mathcal{C}at_\Delta \longrightarrow \mathcal{C}at$

- We just apply the functors $\mathfrak{C}$, π_0 and ev_0 to hom-object of the corresponding categories

Weak Equivalences between simplicial cats

Definition A simplicial functor $F: \mathcal{C} \to \mathcal{D}$ between simplicially enriched cats is called a weak equiv. if it induces

- WE on all hom-simplicial sets (weakly fully faithful)

- an essentially surjective functor
  $$\pi(\mathcal{C}) \longrightarrow \pi(\mathcal{D})$$
  (weakly essentialy surjective)

Simplicial cats and simplicial objects

- Any simplicial cat $\mathcal{C}$ gives rise to a simplicial object $\mathcal{C}_\bullet : \Delta^{op} \to \mathrm{Cat}$ in Cat:

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{C}_1 \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \mathcal{C}_2 \cdots \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \end{tikzcd}$$

- Each of cat $\mathcal{C}_n$ has the same objects as $\underline{\mathcal{C}}$
- Define

$$\mathcal{C}_n(x,y) := \underline{\mathcal{C}}(x,y)_n,$$

i.e. arrows in $\mathcal{C}_n$ are n-simplices in $\underline{\mathcal{C}}(x,y)$
- In particular, $\mathcal{C}_0$ is the underlying cat of $\underline{\mathcal{C}}$

Simplicial cats and simplicial objects

- Conversely, any simplicial object $\mathcal{C}_\bullet : \Delta^{op} \longrightarrow \mathcal{C}at$, s.t.

$$d_i : \mathcal{C}_n \longrightarrow \mathcal{C}_{n-1}, \quad s_i : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$$

are the identity on objects

- Define

$$\underline{\mathcal{C}}(x, y)_n \text{ to be } \mathcal{C}_n(x, y)$$

- The simplicial action will be specified by the use of functors d_i and s_i

Topological vs. simplicial categories

- We have the following lax monoidal Quillen adjunction

$$\begin{tikzcd} sSet \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & Top \arrow[l, shift left, "S"] \end{tikzcd}$$

- It induces with the lax monoidal localization functors

$$h: sSet \longrightarrow Ho(sSet), \quad h: Top \longrightarrow Ho(Top)$$

the change of base adjunction

$\mathcal{H} := Ho(sSet)$

$$\begin{tikzcd} Cat_{sSet} \arrow[r, shift left, "|-|_*"] \arrow[r, phantom, "\perp"] \arrow[dr, "h"'] & Cat_{Top} \arrow[l, shift left, "S_*"] \arrow[d, "h"] \\ & Cat_{\mathcal{H}} \end{tikzcd}$$

$Cat_{\mathcal{H}}$ is the category of small categories enriched over homotopy types

Locally Kan simplicial categories & Bergner's model structure

Definition A simplicial category is *locally Kan* if each of its hom-space is a Kan complex

- There is a cofibrantly generated model structure on sCat

Simplicial cats and simplicial objects

- WE — simplicial functors that descend to $\mathcal{H}$-equivalences
- Fibrant objects — locally Kan simplicial categories
- Generating cofibrations:

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left, "A"] & 1 \arrow[loop right] \end{tikzcd}$$

$2[A]$ is simplicial cat with $0, 1$ as objects

Hom-spaces: $2[A](0,0) = 2[A](1,1) = *$, $\quad 2[A](0,1) = A$

$$2[A](1,0) = \emptyset$$

Locally Kan simplicial categories & Bergner's model structure

Theorem (Bergner) $\exists$ a cofib. gen. mod. struct. on $s\mathcal{C}at$ whose WE are $F: \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, s.t. $hF: \underline{h\mathcal{C}} \to \underline{h\mathcal{D}}$ is an $\mathcal{H}$-equiv; fibrant objects are the locally Kan simpl. cats and whose cofibs are generated by

$$\{\emptyset \longrightarrow *\} \cup \{2[\partial\Delta^n] \longrightarrow 2[\Delta^n]\}_{n\geq 0}$$

Cofibrant simplicial cats & simplicial computads

- An n-arrow $f: a \to b$ in $\mathcal{C}_n$ is just an n-simplex in the simplicial set $\mathcal{C}(a,b)$ for simplicial cat $\mathcal{C}$

- By Eilenberg–Zilber lemma any n-simplex

We say "f has dimension m"

$$f = f' \cdot \alpha$$

non-degen. m-arrow

$\alpha : [n] \to [m]$ is an epi

Cofibrant simplicial cats & simplicial computads

Definition An arrow in an unenriched cat is atomic if it admits no non-trivial factorizations

Definition A cat is freely generated by a reflexive directed graph if each of its arrow may be uniquely expressed as a composite of atomic arrows

Cofibrant simplicial cats & simplicial computads

Definition A simplicial cat $\mathcal{C}: \Delta^{op} \to \mathcal{C}at$ is a simplicial computad if

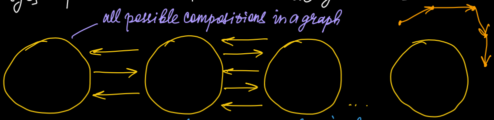
- each $\mathcal{C}_n$ is freely generated
- for each surjection $\alpha: [n] \twoheadrightarrow [m]$ and atomic arrow $f \in \mathcal{C}_m$, the arrow

$$f \cdot \alpha \text{ is atomic in } \mathcal{C}_n$$

Cofibrant simplicial cats & simplicial computads

- A simplicial computad is a simplicial object in Cat, each of whose categories is freely generated on a set of generating arrows that includes the degenerate images of all lower dimensional generators

all possible compositions in a graph



no non-degenerate simplices in dimension >0

Cofibrant simplicial cats & simplicial computads

Lemma The simplicial computads are the cellular cofibrant objects in sCat.
Furthermore, every cofibrant object is cellular, and hence a simplicial computad

Proof • Prove that any cofibrant simplicial category is a simplicial computad

• Check that a retract of a simplicial computad is a simplicial computad

Cat

- A retract $\mathcal{B} \hookrightarrow \mathcal{C} \twoheadrightarrow \mathcal{B}$ in of free category is a free category
- This is it since if $h = fg$ and any two of these are in $\mathcal{B}$, so is the third
- By induction, any arrow in $\mathcal{B}$ is uniquely decomposable into the shortest composites of atomic arrows of $\mathcal{C}$ that lie in $\mathcal{B}$
- So, at each level a retract of simplicial computad is a free category

- Now prove that the degenerate images of atomic arrows in $\mathcal{B}_n$ are atomic in $\mathcal{B}_{n+1}$
- This is clear for atomic arrows in $\mathcal{B}_n$ that are also atomic in $\mathcal{C}_n$
- Suppose that a degenerate image of some atomic arrow in $\mathcal{B}_n$ factors as

$$gf \text{ in } \mathcal{B}_{n+1}$$

- $\mathcal{C}$ is simpl. computad $\Rightarrow$ we have $g'f'$ in $\mathcal{C}_n$ with $g' \mapsto g$, $f' \mapsto f$

- Apply one of the face maps that serves as a retraction of the degeneracy $\rightsquigarrow$ either g or f must map to an identity in $\mathcal{B}_n$
- So, one of g' or f' is an identity $\triangleleft$

A free-forgetful adjunction

$$\begin{tikzcd} F : \mathrm{rDirGph} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Cat} : U \arrow[l, shift left] \end{tikzcd}$$

reflexive directed graphs

- We have the comonad resolution associated to a small cat $\mathcal{A}$

$$\begin{tikzcd}[column sep=huge] FU\mathcal{A} & FUFU\mathcal{A} \arrow[l, shift right=2, "\varepsilon FU"'] \arrow[l, shift left=2, "FU\varepsilon"] & FUFUFU\mathcal{A} \cdots \arrow[l, shift right=4, "\varepsilon FUFU"'] \arrow[l, "FU\varepsilon FU"] \arrow[l, shift left=4, "FUFU\varepsilon"] \\ \end{tikzcd}$$

$$FU\mathcal{A} \xrightarrow{F\eta U} FUFU\mathcal{A} \xrightarrow{F\eta UFU,\ FUF\eta U} FUFUFU\mathcal{A}$$

- Note that this comonad resolution is a simplicial computad $FU_\bullet\mathcal{A}$
- Hence, it is a cofibrant simplicial category by the Lemma above
- $FU\mathcal{A}$ is the free category

$((\bullet \to (\bullet \to) \cdots \to \bullet))$ — a general arrow in $FU_\bullet\mathcal{A}$

$(\bullet \to (\bullet \to) \cdots \to \bullet)$ — an atomic arrow in $FU_\bullet\mathcal{A}$

$FUFU\mathcal{A} \rightsquigarrow ((\cdots \to \to \to \cdots))$

$$(FU)^k\varepsilon(FU)^j \rightsquigarrow \underbrace{(\ldots}_{k\text{ times}}\overbrace{(\ldots}^{j\text{ times}}(\ldots \to\to\to \ldots)\ldots)\ldots)$$

compose arrows and remove these parenthesis

$$FU\ldots FU \rightsquigarrow (\ldots(\ldots(\ldots\to\to\to\ldots)\ldots)\ldots)$$

compose arrows and remove these parenthesis

$F(UF)^k \eta (UF)^j U$ $\leftrightsquigarrow$ double up the parenthesis that are contained in exactly k others

$F \ldots UF\eta F$ $\leftrightsquigarrow$ insert parenthesis around each individual morphism

- Recall, the main task of this activity is to motivate the correct choice of an adjunction

$$\mathfrak{C} : sSet \rightleftarrows sCat$$

A naïve way to construct $\mathfrak{C}$

- A naïve choice for $\mathfrak{C}\Delta^\bullet : \Delta \longrightarrow sCat$

  $[n]$ — a discrete simplicial category

But: the right adjoint will be an ordinary nerve

- $\mathfrak{C}\Delta^\bullet$ is supposed to be a simplicial category that encodes a "homotopy coherent" diagram of shape $[n]$

A homotopy commutative diagrams

Definition A homotopy commutative diagram of shape $\mathcal{A}$ is a map of reflexive directed graphs

a small cat

$$U\mathcal{A} \longrightarrow U\mathcal{C}$$

that defines a functor

$$\mathcal{A} \longrightarrow h\mathcal{C}$$

- A diagram $F: U\mathcal{A} \to U\mathcal{C}$ is homotopy commutative if whenever $h = fg$ in $\mathcal{A}$, Fh and $Fg \cdot Ff$ lie in the same path component of the hom-space

$\mathcal{H}$ homotopy commutative diagrams

- When $\mathcal{C}$ is locally Kan, this is the case when $\exists$ 1-simplices

$$Fh \longrightarrow Fg \cdot Ff$$
&
$$Fg \cdot Ff \longrightarrow Fh$$

$\mathcal{A}$ homotopy coherent diagram

Definition $\mathcal{A}$ homotopy coherent diagram of shape $\mathcal{A}$ is a simplicial functor

$$FU_\bullet\mathcal{A} \longrightarrow \mathcal{C}$$

- By means of the map

$$U\mathcal{A} \xrightarrow{U\eta} UFU\mathcal{A}$$

One can construct the homotopy commutative diagram

$$U\mathcal{A} \xrightarrow{U\eta} UFU\mathcal{A} \xrightarrow{U(FU_\bullet\mathcal{A}\to\mathcal{C})} U\mathcal{C}$$

A homotopy coherent natural transform

Definition A hom. coh. nat. tr. is a homotopy coherent diagram of shape

$$\mathcal{A} \times \mathbb{2},$$

i.e., a simplicial functor

$$\mathfrak{C}\,(\mathcal{A} \times \mathbb{2}) \longrightarrow \underline{\mathcal{C}}$$

The main motivation was...

- Given a commutative diagram $F : \mathcal{A} \to \mathcal{C}$
- Is it possible to form a new diagram in which each object is replaced by a specified homotopy equivalent one?
- Given $\alpha : F \Rightarrow G$, is it possible to replace maps α_n with homotopic ones?
- The answer is no in general, but...

The main motivation was...

Proposition (Cordier–Porter) Given a homotopy coherent diagram $F: \mathcal{A} \to \underline{\mathcal{C}}$ in a locally Kan simplicial category and a family of homotopy equivalences

$$Fa \longrightarrow Ga$$

This data extends to a homotopy coherent diagram

$$G: \mathcal{A} \longrightarrow \underline{\mathcal{C}}$$

and homotopy coherent map

$$F \Rightarrow G$$

A fruitful example

- Consider the category $[3]$

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[rr, bend right, "j"'] \arrow[rrr, bend right=60, "\ell"'] & 1 \arrow[r, "g"] \arrow[rr, bend right, "k"'] & 2 \arrow[r, "h"] & 3 \end{tikzcd}$$

Describe the hom-space in $FU_\bullet[3](0,3)$

- The vertices of $FU_\bullet[3](0,3)$ are the paths of edges:

$$\ell,\ kf,\ hj \text{ and } hgf \qquad (*)$$

- To obtain the 1-simplices one should write one pair of parenthesis for each in $(*)$

A fruitful example

- The non-degenerate simplices fit into the diagram

$$\begin{tikzcd}
\ell \arrow[r, "(kf)"] \arrow[d, "(hj)"'] \arrow[dr, "(hgf)"] \arrow[dr, phantom, "((hg)(f))", shift left=3] \arrow[dr, phantom, "((h)(gf))"', shift right=3] & kf \arrow[d, "(hg)(f)"] \\
hj \arrow[r, "(h)(gf)"'] & hgf
\end{tikzcd}$$

Question What does $\mathfrak{F}\mathfrak{U}_\bullet[3](0,3)$ look like?

The desired definition of $\mathfrak{C}$

Definition $\mathfrak{C}[\Delta^n] := FU_\bullet[n]$

That is, $\mathfrak{C}[\Delta^n]$ is a homotopy coherent diagram of shape $[n]$

Lemma The cat Cat_Δ admits all small colimits. Hence, there exists a unique colimit-preserving functor

$$\mathfrak{C}[-] : sSet \to Cat_\Delta$$

which sends Δ^n to $\mathfrak{C}[\Delta^n]$ already defined

▷ Consider an appropriate left Kan extension ◁

Another definition of $\mathfrak{C}$

Definition Let J be a finite non-empty linearly ordered set, and let i, j be elements of this set. We let

$$P_{i,j} := \{ I \subseteq J \mid i, j \in I \text{ and } k \in I \Rightarrow i \leq k \leq j \},$$

i.e., $P_{i,j}$ consists of all subsets of $[i,j] \subseteq J$ which contain i and j

- $P_{i,j}$ is a poset

Another definition of $\mathfrak{C}$

- For $i \le j \le k$ in J, there is a canonical map of posets

$$P_{i,j} \times P_{j,k} \longrightarrow P_{i,k}$$
$$(I, I') \longmapsto I \cup I'$$

Definition Objects of $\mathfrak{C}[\Delta^J]$ — the elements of J

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) := \begin{cases} \emptyset, & \text{if } i > j \\ N(P_{i,j}), & \text{if } i \le j \end{cases}$$

Composition is defined via the previous observation

Some simple properties

- $\forall n \geq 1 \quad N(P_{0,n}) \cong (\Delta^1)^{n-1}, \quad P_{i,j} \cong P_{0,j-i}$

- For $i \leq j$, the simplicial set $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j)$ is contractible

- There is a unique isomorphism

$$\pi(\mathfrak{C}[\Delta^n]) \cong [n]$$

which is identity on objects

- By adjunction $\mathfrak{C}[\Delta^n] \xrightarrow{\cong} c[n]$ a WE of simplicial cats

Non-properties of $\mathfrak{C}$

- $\mathfrak{C}$ does not preserve products

- The space $\mathfrak{C}(x, y)$ are essentially never Kan complexes or even quasi-categories

The homotopy coherent nerve

Definition Let $\mathcal{C}$ be a simplicially enriched category. Then its simplicial nerve (or the homotopy-coherent nerve) is the simplicial set

$$\mathfrak{N}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

- If $\mathcal{C}$ is an ordinary cat

$$N(\mathcal{C}) \cong \mathfrak{N}(c\mathcal{C})$$

a constant enrichment

$\triangleright$ $\mathfrak{N}(c\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], c\mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}(\pi(\mathfrak{C}[\Delta^n]), \mathcal{C})$

$\cong \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) = N(\mathcal{C})_n$ $\triangleleft$

An adjunction and a Quillen equivalence

Theorem The pair $\mathfrak{C} \dashv \mathcal{R}$ forms a Quillen equivalence between Joyal's model structure for quasi-categories and Bergner's model structure for simplicial categories

- So, the homotopy coherent nerve of a locally Kan simplicial category is a quasi-category

a source of examples

An adjunction and a Quillen equivalence

Lurie's definition of a WE in the Joyal's model structure

Corollary The map $X \to Y$ of simplicial sets is a categorical equivalence $\iff$ the induced functor

$$h\mathfrak{C}X \longrightarrow h\mathfrak{C}Y$$

is an equivalence of $\mathcal{H}$-categories

Some fruitful results on $\mathfrak{C} \dashv \mathfrak{N}$

Theorem Let A be any small cat. Then

$$\mathfrak{C} N A \cong F U_{\bullet} A$$

Theorem If $\mathcal{C}$ is a locally Kan simplicial cat, then

$$\mathfrak{N}\mathcal{C} \text{ is a quasi-cat}$$

Example • For a simplicial model cat $\mathcal{M}$ we have a right Quillen bifunctor $\mathcal{M}^{op} \times \mathcal{M} \longrightarrow sSet$

• The category $\mathcal{M}_{QR}$ is locally Kan

• So, $\mathfrak{N}\mathcal{M}_{QR}$ is the quasi-cat, and $h\mathfrak{N}\mathcal{M}_{QR} \cong h\mathcal{M}$

The original motivation by Boardman & Vogt

- Homotopy coherent diagrams and natural transformations assemble into a quasi-cat
- This was an example that motivated the original motivation by Boardman & Vogt:

$\forall$ small cat $\mathcal{A}$ and $\forall$ locally Kan simplicial cat $\underline{\mathcal{C}}$

$\mathfrak{C}N(\mathcal{A}\times[n]) \longrightarrow \underline{\mathcal{C}}$ ← an n-simplex of some simpl. set

Vertices — homotopy coherent diagrams $\mathcal{A} \to \underline{\mathcal{C}}$

Edges — homotopy coherent natural transf. between $\mathcal{A} \to \underline{\mathcal{C}}$

The original motivation by Boardman & Vogt

- By adjunction, n-simplices are

$$\begin{array}{c} N(\mathcal{A}\times[n]) \longrightarrow \mathcal{KC} \\ \| \\ N\mathcal{A}\times\Delta^n \end{array}$$

- Once again, by adjunction the n-simpl. are iso to

$$\Delta^n \longrightarrow (\mathcal{KC})^{N\mathcal{A}}$$

- So, by Yoneda, our simplicial set is isomorphic to

$$(\mathcal{KC})^{N\mathcal{A}}$$

$\mathcal{KC}$ a quasi-cat $\Rightarrow$ it is a quasi-cat

Thank you!