<u>Plan:</u>

1. Reedy mod structure in Top } (Emily Riehl, Chapter 14)
2. —"— simpl. mod. cat.
3. Reedy cats Hirschhorn's Lemma } (Hirschhorn, Chapter 15)
4. Reedy mod structure and Kan's Theorem
5. Applications (The next talk...)

$X : \Delta^{op} \to \mathrm{Top}$

$$sk_1|X| \to sk_2|X| \to sk_3|X| \to \dots$$

$$sk_n|X| = \Delta^{\bullet}_{\leq n} \otimes_{\Delta^{op}_{\leq n}} X_{\leq n}$$

$sk_n|X|$

$$\begin{tikzcd}
{|\Delta^n| \times \mathcal{L}_n X \sqcup |\partial\Delta^n| \times X_n} \arrow[r] \arrow[d, "\hat{i}_n \hat{\times} \ell_n"'] & sk_{n-1}|X| \arrow[d] \\
{|\Delta^n| \times X_n} \arrow[r] & sk_n|X| \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$i_n : |\partial\Delta^n| \hookrightarrow |\Delta^n|$ is a cofib in Top

$\ell_n : \underset{\text{latching object}}{\underbrace{\mathcal{L}_n X}} \to X_n$ — latching map is a cofib.

So, $|-|$ preserves WE in this case

<u>Simplicial model cat setting</u>

$X : \Delta^{op} \to \mathcal{M}$

$M_n X$ with a map $m_n : X_n \longrightarrow M_n X$ – "boundary data"

$\lim^{\Delta^n} X = X_n$ $\qquad$ $X : \Delta^{op} \to \mathcal{M}$

$\quad \|$

$\int_{m \in \Delta} \mathcal{M}(\Delta^n(m), X(m))$ $\qquad$ $\Delta^\bullet : \Delta \to \mathcal{M}^{\Delta^{op}}$

Def. $M_n X := \lim^{\partial\Delta^n} X$

$$\begin{array}{cccc} m_n : & X_n & \longrightarrow & M_n X \\ & \| & & \| \\ & \lim^{\Delta^n} X & \longrightarrow & \lim^{\partial\Delta^n} X \end{array}$$

Example.] X is a simpl. set

$\lim^{\partial\Delta^n} X$ is $\{\partial\Delta^n \to X\}$

As $W : \mathcal{D} \to Set$, $F : \mathcal{D} \to Set$

$\lim^{W} F = F^{W} = \{W \Rightarrow F\}$ (easy exercise)

$$\partial\Delta^n \cong \operatorname{colim}\left(\coprod_{[n-2] \rightarrowtail [n]} \Delta^{n-2} \underset{[n-2] \rightarrowtail [n-1]}{\overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}}} \coprod_{[n-1] \rightarrowtail [n]} \Delta^{n-1} \right)$$

Lemma (exercise on ninja-Yoneda Lemma)

$$\lim^{\operatorname{colim} W_i} F \cong \lim\left(\lim^{W_i} F\right)_{I}$$

$W : I \to Cat(\mathcal{C}, Set)$

$\quad i \longmapsto W_i$

$$\lim^{\partial\Delta^n} X \cong \int_{m\in\Delta} \mathrm{Power}\Big(\coprod_{[n-2]\rightarrowtail[n]} \Delta^{n-2} \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \coprod_{[n-1]\rightarrowtail[n]} \Delta^{n-1}\Big) \cong$$

$(-,*)$-contrav.

$$\cong \int_{m\in\Delta} \mathrm{Power}\Big(\coprod_{[n-1]\rightarrowtail[n]} \Delta^{n-1}_m, X_m\Big) \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \int_{m\in\Delta} \mathrm{Power}\Big(\coprod_{[n-2]\rightarrowtail[n]} \Delta^{n-2}_m, X_m\Big) \cong$$

$$\cong \prod_{[n-1]\rightarrowtail[n]} \int_{m\in\Delta} \mathrm{Power}(\Delta^{n-1}_m, X_m) \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \prod_{[n-2]\rightarrowtail[n]} \int_{m\in\Delta} \mathrm{Power}(\Delta^{n-2}_m, X_m) \cong$$

$$\overset{\text{L}}{\cong} \prod_{[n-1]\rightarrowtail[n]} X_{n-1} \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \prod_{[n-2]\rightarrowtail[n]} X_{n-2}$$

$$\boxed{M_n X \cong \lim\Big(\prod_{[n-1]\rightarrowtail[n]} X_{n-1} \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \prod_{[n-2]\rightarrowtail[n]} X_{n-2}\Big)}$$

<u>Def.</u> $L_n X := \mathrm{colim}^{\partial\Delta_n} X$

$\ell_n : L_n X \to X_n$ which is induced by $\partial\Delta_n \to \Delta([n],-)$

<u>Reedy cats and Reedy model structures</u>

<u>Def.</u> A Reedy cat is a small cat $\mathcal{D}$ equipped with

(i) a degree function $\deg : \mathcal{D} \to \mathbb{Z}_{\geq 0}$

(ii) a wide subcat $\vec{\mathcal{D}}$ whose non-identity morph.s strictly raise degree

(iii) — " — $\overleftarrow{\mathcal{D}}$ — " — lower degree

$$f = \underset{\overleftarrow{\mathcal{D}}}{\underset{\uparrow}{\vec{g}}} \cdot \underset{\overleftarrow{\mathcal{D}}}{\underset{\uparrow}{\overleftarrow{g}}} \qquad \forall f \in \mathcal{D}_1$$

Example. The cat Δ is a Reedy cat, $\deg([n]) = n$

$\overleftarrow{\Delta}$ is formed by epimorphisms

$\vec{\Delta}$ — " — monomorphisms

For $X \in \mathcal{M}^{\mathcal{D}}$, $X_d = X^d =: X(d)$

$$\mathcal{L}^d X \xrightarrow{l_d} X^d \xrightarrow{m_d} M^d X$$

$\mathcal{D}$ is Reedy cat

$\mathcal{M}$ is some model cat

Def. The dth latching object of $X \in \mathcal{M}^{\mathcal{D}}$

$$\mathcal{L}^d X := \operatorname{colim}\left(\vec{\mathcal{D}}_{<n}/d \xrightarrow{U} \mathcal{D} \xrightarrow{X} \mathcal{M}\right)$$

$\deg(d) = n$

$$\mathcal{M}^d X := \lim\left(d/\overleftarrow{\mathcal{D}}_{<n} \xrightarrow{U} \mathcal{D} \xrightarrow{X} \mathcal{M}\right)$$

$$\vec{\mathcal{D}}_{<n}/d =: \partial(\vec{\mathcal{D}}/d)$$

Example $F^0\mathcal{D}$ is a descr. cat

Consider the problem of extending some functor

$X : F^{n-1}\mathcal{D} \to \mathcal{M} \quad \rightsquigarrow \quad X : F^n\mathcal{D} \to \mathcal{M}$

$X : F^0\mathcal{D} \longrightarrow \mathcal{M}$

$\forall \alpha \in \mathcal{D} \rightsquigarrow X_\alpha$

- On the $(n-1)$ th step we choose X_α in $\mathcal{M}$

$\forall \alpha \in \mathcal{D}$, $\deg(\alpha) = n$

$$\begin{tikzcd}
\beta \arrow[r] \arrow[dr] & \beta' \arrow[d] & \in F^{n-1}\mathcal{D} \\
& \alpha &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
X_\beta \arrow[r] \arrow[dr] & X_{\beta'} \arrow[d] \\
& X_\alpha
\end{tikzcd}$$

$$\begin{tikzcd}
F^{n-1}\mathcal{D} \arrow[r, "X"] \arrow[d, hook, "I^n"'] \arrow[r, phantom, shift right=4, "\Downarrow"] & \mathcal{M} \\
F^n\mathcal{D} \arrow[ur, dashed] &
\end{tikzcd}$$

$$\mathrm{Lan}_{I^n} X(\alpha) = \operatorname*{colim}_{I^n/\alpha} \mathcal{D}(I_n -, \alpha) X \cong \operatorname*{colim}_{I^n/\alpha} XU$$

$$\begin{tikzcd}
& \alpha \arrow[dl] \arrow[dr] & \\
\gamma \arrow[rr] & & \gamma'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& X_\alpha \arrow[dl] \arrow[dr] & \\
X_\gamma \arrow[rr] & & X_{\gamma'}
\end{tikzcd}$$

$$\begin{tikzcd}
& \alpha \arrow[dr] & \\
\beta \arrow[ur] \arrow[rr] & & \gamma
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& X_\alpha \arrow[dr] & \\
X_\beta \arrow[ur] \arrow[rr] & & X_\gamma
\end{tikzcd}$$

$$\operatorname*{colim}_{I^n/\alpha} X \longrightarrow \lim_{\alpha/I^n} X$$

$$\operatorname*{colim}_{I^n/\alpha} X \longrightarrow X_\alpha \longrightarrow \lim_{\alpha/I^n} X$$

Theorem. $\exists \mathcal{D}$ is a Reedy
$\exists \mathcal{M}$ is a bicomplete cat

$n>0$, $X : F^{n-1}\mathcal{D} \to \mathcal{M}$

If $\forall \alpha \in \mathcal{D}$ $\deg(\alpha) = n$ we choose an object X_α and choose a factorization

$$\operatorname*{colim}_{I^n/\alpha} X \longrightarrow X_\alpha \longrightarrow \lim_{\alpha/I^n} X,$$

then this uniq. determ. an extension $X : F^n\mathcal{D} \to \mathcal{M}$

Proof: $\alpha \longrightarrow \alpha'$, $\alpha, \alpha' \in F^n\mathcal{D}$

- $\alpha \to \alpha' = \alpha \xrightarrow{\overleftarrow{g}} \beta \xrightarrow{\overrightarrow{g}} \alpha'$

We define $X_\alpha \to X_{\alpha'} = X_\alpha \xrightarrow{X(\overleftarrow{g})} X_\beta \xrightarrow{X(\overrightarrow{g})} X_{\alpha'}$

$\alpha \to \alpha' \to \alpha'' \rightsquigarrow$
$$\begin{tikzcd} & X_\alpha \arrow[dl] \arrow[dr] & \\ X_{\alpha'} \arrow[rr] & & X_{\alpha''} \end{tikzcd}$$

See Hirschhorn's Book for details $\triangleleft$

Prop. Let $\mathcal{D}$ be a Reedy cat
$\alpha \in \mathcal{D}_0$, $\deg(\alpha) = n$, $I^n : F^{n-1}\mathcal{D} \hookrightarrow F^n\mathcal{D}$. Then

(1) The latch. cat $\overrightarrow{\mathcal{D}}_{<n}/\alpha$ is a final subcat of I^n/α; $(F^n\mathcal{D} = \mathcal{D}_{\leq n})$

(2) —//—

Corollary. $\operatorname*{colim}_{I^n/\alpha} X \cong \mathcal{L}_\alpha X \cong \operatorname*{colim}_{\overrightarrow{\mathcal{D}}_{n}/\alpha} X \; (\cong \operatorname{colim} X \circ I^n)$

$\mathcal{I}^n : \overrightarrow{\mathcal{D}}_{<n/\alpha} \hookrightarrow I^n/\alpha$

Maps betw. diagrams $M^{\mathcal{D}}$

$f : X \to Y, \quad X, Y \in M^{\mathcal{D}}$

- $f : X|_{F^0\mathcal{D}} \to Y|_{F^0\mathcal{D}}$

$f : X_\alpha \to Y_\alpha \quad \forall \alpha \in \mathcal{D}$

- If we have $f : X|_{F^{n-1}\mathcal{D}} \to Y|_{F^{n-1}\mathcal{D}}$

$$\begin{tikzcd}
L_\alpha X \cong \operatorname{colim}_{I^n/\alpha} X \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d, dashed] & \lim_{\alpha/I^n} X \cong M_\alpha X \arrow[d] \\
L_\alpha Y \cong \operatorname{colim}_{I^n/\alpha} Y \arrow[r] & Y_\alpha \arrow[r] & \lim_{\alpha/I^n} Y \cong M_\alpha Y
\end{tikzcd}$$

The extensions of f to the n-filtration of $\mathcal{D}$ correspond to a choise, for every object α, $\deg(\alpha) = n$, of a dotted arrow that makes both squares commute

Lemma (Hirschhorn's lemma). If A, B, X, Y are $\mathcal{D}$-diagrams in $\mathcal{M}$ and

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

h is defined on $F^{n-1}B$

Then $\forall \alpha \in \mathcal{D}_0$ $\deg(\alpha) = n$ we have

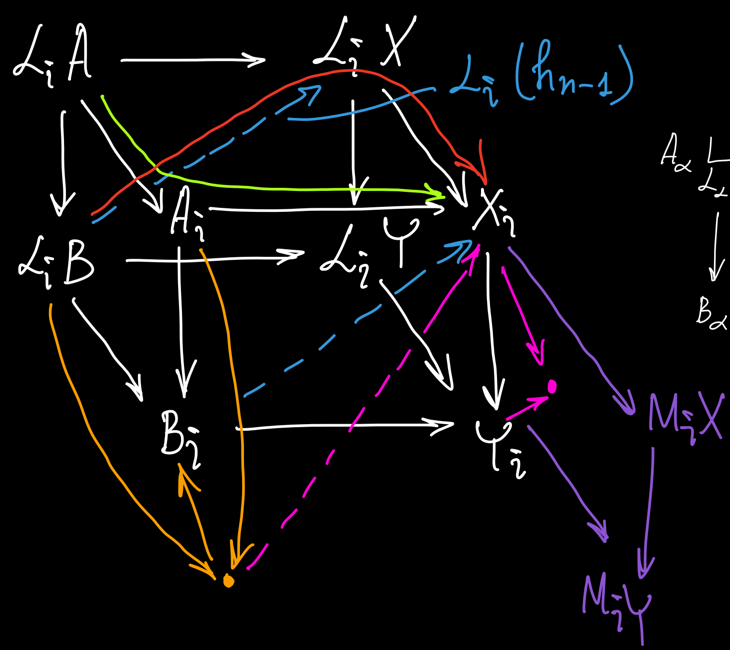
$$\begin{tikzcd} A_\alpha \coprod_{L_\alpha A} L_\alpha B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B_\alpha \arrow[r] \arrow[ur, dashed, "H"] & Y_\alpha \times_{M_\alpha Y} M_\alpha X \end{tikzcd}$$

Furthermore, there is a map $H: B_\alpha \to X_\alpha$ $\forall \alpha$ $\deg(\alpha) = n$

$\iff$ h can be extended over the restriction of B to $F^n\mathcal{D}$

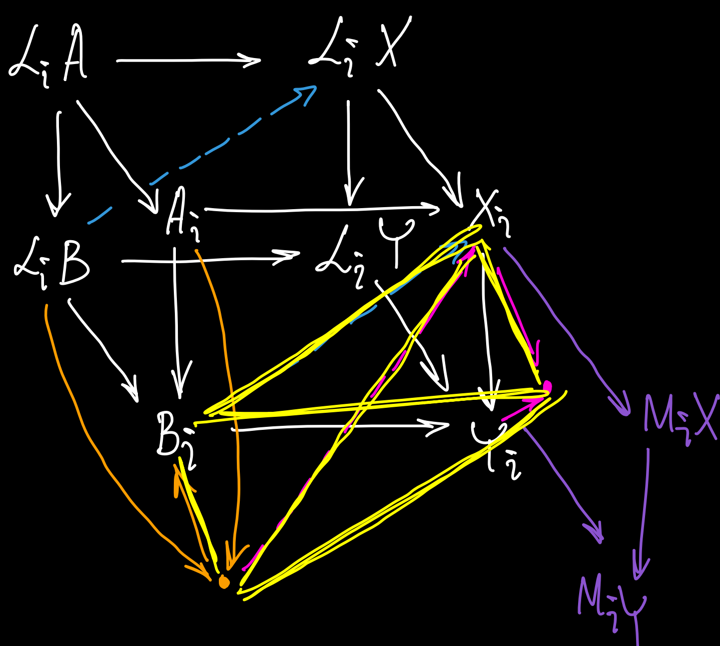
Proof: $2 \Rightarrow 1$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$



$$\begin{tikzcd}
A_\alpha \coprod_{\mathcal{L}_\alpha A} \mathcal{L}_\alpha B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\
B_\alpha \arrow[r] \arrow[ur, dashed, "H"] & Y_\alpha \times_{M_\alpha Y} M_\alpha X
\end{tikzcd}$$

$\underline{H = h}$



Def. $\mathcal{D}$ – Reedy cat, $\mathcal{M}$ – model cat

$f : X \to Y \in (\mathcal{M}^{\mathcal{D}})_1$

(1) If $\alpha \in \mathcal{D}_0$, then the relative matching map is the map $X_\alpha \coprod_{\mathcal{L}_\alpha X} \mathcal{L}_\alpha Y \to Y_\alpha$

(2) — " — $X_\alpha \longrightarrow Y_\alpha \times_{M_\alpha Y} M_\alpha X$

Def. (1) $f: X \to Y$ is Reedy WE if $\forall \alpha \in \mathcal{D}_0$
$f_\alpha : X_\alpha \longrightarrow Y_\alpha$ is a WE in $\mathcal{M}$

(2) $f: X \to Y$ is a Reedy cofib if $\forall \alpha \in \mathcal{D}$

$$X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$$

is a cofibr. in $\mathcal{M}$

(3) $f: X \to Y$ is Reedy fibration: $\forall \alpha \in \mathcal{D}_0$

$$X_\alpha \longrightarrow Y_\alpha \times_{M_\alpha Y} M_\alpha X$$

is a fibr. in $\mathcal{M}$

Theorem (D. Kan). (1) The cat $\mathcal{M}^{\mathcal{D}}$ is a model cat with the Reedy WE, Reedy cofibs, Reedy fibs
(2) If $\mathcal{M}$ is a left proper, then — " —

Example. $\mathcal{M}^{\Delta^{op}}$ has a Reedy mod structure

Lemma. $f: X \to Y$, $\alpha \in \mathcal{D}_0$
S is a class of maps in $\mathcal{M}$

(1) If $\forall \beta \in \mathcal{D}_0$ $\deg(\beta) < \deg(\alpha)$

$$X_\beta \bigsqcup_{\mathcal{L}_\beta X} \mathcal{L}_\beta Y \to Y_\beta$$

has the LLP w.r.to S, then

$$\mathcal{L}_\alpha X \to \mathcal{L}_\alpha Y$$

has LLP w.r. S

(2) — // —

Proof: $F^0 \partial(\vec{\mathcal{D}}/\alpha)$, $F^{\deg(\alpha)-1}\partial(\vec{\mathcal{D}}/\alpha) = \partial(\vec{\mathcal{D}}/\alpha)$

$$\begin{tikzcd} \mathcal{L}_\alpha X \arrow[r] \arrow[d] & E \arrow[d] \\ \mathcal{L}_\alpha Y \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

$$\operatorname*{colim}_{F^k\partial(\vec{\mathcal{D}}/\alpha)} Y$$

$\cdot \deg(\beta) = 0$, $(\beta \to \alpha) \in \vec{\mathcal{D}}/\alpha$

$$\mathcal{L}_\beta X = \mathcal{L}_\beta Y \Rightarrow (X_\beta \to Y_\beta) = \left(X_\beta \bigsqcup_{\mathcal{L}_\beta} \mathcal{L}_\beta Y \to Y_\beta\right)$$

$$\begin{tikzcd} X_\beta \arrow[r] \arrow[d] & E \arrow[d] \\ Y_\beta \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{L}_\beta X \arrow[r] \arrow[d] & E \arrow[d] \\ \mathcal{L}_\beta Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The inductive step: $\exists\, 0 < k < \deg(\alpha)$

h has been defined on $\operatorname*{colim}_{F^{k-1}\partial(\vec{\mathcal{D}}/\alpha)} Y$

Let $(\beta \to \alpha) \in \partial(\overrightarrow{\mathcal{D}}/\alpha)$, s.t. $\deg(\beta) = k$

$$\partial(\overrightarrow{\mathcal{D}}/\beta) \longrightarrow F^{k-1}\partial(\overrightarrow{\mathcal{D}}/\alpha)$$

So, it defines the map h on $L_\beta Y$

$$\begin{tikzcd} X_\beta \coprod_{L_\beta X} L_\beta Y \arrow[r] \arrow[d] & E \arrow[d] \\ Y_\beta \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

by conditions

$$\begin{tikzcd} L_\alpha X \arrow[r] \arrow[d] & E \arrow[d] \\ L_\alpha Y \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

by Hirschhorn's Lemma the lifting exists

$\iff$ the other lifting exists in

$$\begin{tikzcd} X_\beta \arrow[r] \arrow[d] & E \arrow[d] \\ Y_\beta \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

$\deg(\beta) < \deg(\alpha)$

by inductive hypothesis

$\triangleleft$