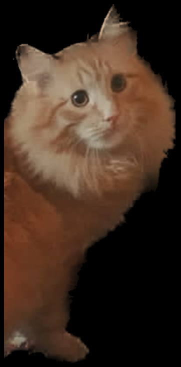


Comma ∞-cat

Motivation

- Universal properties of (co-)limits & adjunctions

  (co-)limits: by means of cones

  adjunctions: by means of bijection between some hom-sets

- Present the equivalent definitions in terms of comma ∞-cats

- Comma ∞-cats will be preserved by functors of ∞-cosmoi

Motivation

- Encoding limits, colimits & adjunctions as equivalences between comma ∞-cats, ones are preserved, reflected & created by weak equivances of ∞-cosmoi

- As a consequence, these notions are invariant under change of models between quasi-cats, complete Segal spaces, Segal cats and naturally marked simplicial sets

Recall: a quasi-categorical setting, smothering functors

Q — quasi-cat

Then its homotopy cat hQ has:

- elements of Q as its objects
- homotopy classes of 1-simplices of Q as its arrows

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

$\exists$ 2-simplex with the outer edge being degenerate

$$\begin{tikzcd}
& \bullet \arrow[dr, equal] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "g"'] & & \bullet
\end{tikzcd}$$

- a composition relation

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet
\end{tikzcd}$$

$gf \sim h$ in hQ

- Let $\mathcal{I}$ be a 1-cat

In general,

$$h(\mathcal{Q}^{\mathcal{I}}) \not\simeq (h\mathcal{Q})^{\mathcal{I}}$$

But it is so when $\mathcal{I}$ is a set

- $h(\mathcal{Q}^{\mathcal{I}})$ – homotopy coherent diagrams of shape $\mathcal{I}$ in $\mathcal{Q}$
- $h(\mathcal{Q})^{\mathcal{I}}$ – homotopy coherent diagrams

A canonical comparison functor

$$h(\mathcal{Q}^{\mathcal{I}}) \longrightarrow (h\mathcal{Q})^{\mathcal{I}}$$

$\mathcal{Q}^{\mathcal{I}} \times \mathcal{I} \xrightarrow{ev} \mathcal{Q} \xrightarrow{h} h\mathcal{Q} \rightsquigarrow$ transpose

In particular, $\mathcal{I} = 2 := N(2) = \Delta[1]$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

Lemma. The canonical functor

$$h(Q^{2}) \longrightarrow (hQ)^{2}$$

is smothering meaning that it is

- surjective on objects
- full
- conservative, i.e., reflects invertibility of morphisms

but not necessarily injective nor faithful

Once again,

Def (smothering functor) A functor $f:\overset{\infty\text{-cats}}{A\to B}$ is smothering $\iff$ it has the right lifting property with respect to the set of functors:

$$\left\{ \begin{array}{c}\varnothing\\ \big\downarrow\\ \mathbb{1}\end{array}\;,\quad \begin{array}{c}\mathbb{1}+\mathbb{1}\\ \big\downarrow\\ \mathbb{2}\end{array}\;,\quad \begin{array}{c}\mathbb{2}\\ \big\downarrow\\ \mathbb{I}\end{array} \right\}$$

$$\mathbb{1} = N\big(\overset{\circlearrowleft}{\bullet}\big) \qquad \mathbb{2} = N(\bullet \longrightarrow \bullet) \qquad \mathbb{I} = N\big(\bullet \underset{\sim}{\overset{\sim}{\rightleftarrows}} \bullet\big)$$

Lemma (smothering fibers). Each fiber of a smothering functor is a nonempty connected groupoid

$f : A \to B$ – smothering

$$\begin{tikzcd} A_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "b"] & B \end{tikzcd}$$

Lemma. $\mathcal{J}$ – 1-cat, free on a reflexive directed graph

Q – quasi-cat

Then

$$h(Q^{\mathcal{J}}) \longrightarrow (hQ)^{\mathcal{J}}$$

is smothering

Lemma. Let p be an isofibration

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The canonical functor

$$h(A \times_B E) \longrightarrow hA \times_{hB} hE$$

is smothering

Lemma. For any tower of isofibrations between quasi-cats

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the functor

$$h(\lim_n E_n) \longrightarrow \lim_n hE_n$$

is smothering

Lemma. $\forall$ cospan $C \xrightarrow{g} A \xleftarrow{f} f$ between quasi-cats

$$\begin{tikzcd}
\mathrm{Hom}_A(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, "{(\mathrm{cod},\ \mathrm{dom})}"] \\
C \times B \arrow[r, "g \times f"'] & A \times A
\end{tikzcd}$$

The functor $h\mathrm{Hom}_A(f,g) \longrightarrow \mathrm{Hom}_{hA}(hf, hg)$ is smothering

These smothering functors express weak universal properties of arrow, pullback & comma constructions in the homotopy 2-cat of any ∞-cosmos

∞-categories of arrows

- The previous constructions motivate the following ones

- An element of an ∞-cat $a : 1 \longrightarrow A$

$\iff$ the vertices of $\underbrace{\mathrm{Fun}(1, A)}_{\text{the underlying quasi-cat}}$

Def (arrow ∞-cat) Let A be an ∞-cat

The ∞-cat of arrows in A is A^2 together with the isofibration

$$A^2 := A^{\Delta[1]} \xrightarrow{(p_1, p_0)} \mathrel{\!\!\to\!\!\!\!\to} A^{\partial\Delta[1]} \cong A \times A$$

induced by $\partial\Delta^1 \hookrightarrow \Delta[1]$ $\qquad 0 : 1 \hookrightarrow 2, \quad 1 : 1 \hookrightarrow 2$

The ∞-cat comes with a canonical 2-cell

Lemma (generic arrow) $\forall$ ∞-cat A, the ∞-cat of arrows A^2 comes equipped with a canonical 2-cell

$$(\ast) \qquad \begin{tikzcd} A^2 \arrow[r, bend left=40, "p_0"{name=U}] \arrow[r, bend right=40, "p_1"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "k"] \end{tikzcd}$$

the generic arrow with codomain A

Proof:

- $\mathrm{Fun}(X, \underbrace{A^2}_{\text{simplicial cotensor}}) \cong \mathrm{Fun}(X, A)^2$ — a strict universal property of A^2
- By the Yoneda lemma $(\ast)$ is encoded by the image of identity

$(\mathrm{id}: A^2 \to A^2) \longmapsto$ an elem. of $\mathrm{Fun}(A^2, A)^2$ at the repr. obj.

a 1-simplex in $\mathrm{Fun}(A^2, A)$

it represents a 2-cell k

- What about a 2-cell k?

  Verify that source & target of k are the domain evaluation and codomain evaluation functors

-

$\mathbb{1}+\mathbb{1} \hookrightarrow \mathbb{2}$

$$\begin{tikzcd}
\mathrm{Fun}(X, A^{\mathbb{2}}) \arrow[r, "\cong"] \arrow[d, "(p_1, p_0)_*"'] & \mathrm{Fun}(X, A)^{\mathbb{2}} \arrow[d, "(\mathrm{cod}, \mathrm{dom})"] \\
\mathrm{Fun}(X, A \times A) \arrow[r, "\cong"] & \mathrm{Fun}(X, A) \times \mathrm{Fun}(X, A)
\end{tikzcd}$$

$A^{\mathbb{1}+\mathbb{1}} \cong A \times A$ $\qquad$ $\mathrm{Fun}(X, A)^{\mathbb{1}+\mathbb{1}} \cong \mathrm{Fun}(X, A) \times \mathrm{Fun}(X, A)$

- From the commutativity of the square, we are done $\triangleleft$

By the Yoneda lemma again

$$h\mathrm{Fun}(X, A^{2}) \longrightarrow h\mathrm{Fun}(X, A)^{2}$$

- This is not a nat. iso, nor a nat. equivalence of cats

However:

Prop. (the weak universal property of the arrow ∞-cat)

The generic arrow
$$\begin{tikzcd} A^{2} \arrow[r, bend left=40, "p_0"] \arrow[r, bend right=40, "p_1"'] \arrow[r, phantom, "\Downarrow k"] & A \end{tikzcd}$$
has a weak univ. prop. in the homotopy 2-cat given by 3 operations

(i) 1-cell induction:

Given α

$$\begin{tikzcd} X \arrow[d, bend right=50, "t"'] \arrow[d, bend left=50, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\ A \end{tikzcd} = \begin{tikzcd} X \arrow[d, dashed, "\ulcorner\alpha\urcorner"] \arrow[dd, bend right=70, "t"'] \arrow[dd, bend left=70, "s"] \\ A^{2} \arrow[d, bend right=50, "p_1"'] \arrow[d, bend left=50, "p_0"] \arrow[d, phantom, "\overset{k}{\Leftarrow}"] \\ A \end{tikzcd}$$

Then $\exists \ulcorner\alpha\urcorner : X \to A^{2}$, s.t.

$s = p_0\ulcorner\alpha\urcorner$, $t = p_1\ulcorner\alpha\urcorner$ & $\alpha = k\ulcorner\alpha\urcorner$

(ii) 2-cell induction: Given functors $a, a' : X \longrightarrow A^2$ & natural transformations τ_1 & τ_0 s.t.

$$\begin{tikzcd}
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] & & & & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
A^2 \arrow[dr, "p_1"'] & & A^2 \arrow[dl, bend right=20, "p_1"', ""{name=P1}] \arrow[dl, bend left=40, "p_0", ""{name=P0, below}] \arrow[ll, Rightarrow, shorten=12mm, "\tau_1"'] & = & A^2 \arrow[dr, bend left=30, "p_0", ""{name=Q0, below}] \arrow[dr, bend right=20, "p_1"', ""{name=Q1}] & & A^2 \arrow[dl, "p_0"] \arrow[ll, Rightarrow, shorten=12mm, "\tau_0"'] \\
 & A & & & & A &
\arrow[Rightarrow, from=P0, to=P1, "k"']
\arrow[Rightarrow, from=Q0, to=Q1, "k"']
\end{tikzcd}$$

$\exists$ a natural transformation $\tau : a \Rightarrow a'$

$$p_1 \tau = \tau_1 \quad \& \quad p_0 \tau = \tau_0$$

(iii) 2-cell conservativity: $\forall$ natural transformation

$$\begin{tikzcd}
X \arrow[r, bend left=40, "a", ""{name=U, below}] \arrow[r, bend right=40, "a'"', ""{name=D}] & A^2
\arrow[Rightarrow, from=U, to=D, "\tau"]
\end{tikzcd}$$

if both $p_1\tau$ & $p_0\tau$ – iso then τ is iso too

Proof: Let $Q = \mathrm{Fun}(X, A)$ in the setting of $h(Q^2) \longrightarrow (hQ)^2$ to be smothering

$$\begin{tikzcd}
h\mathrm{Fun}(X, A^2) \arrow[rr] \arrow[dr, "{(p_{1*}, p_{0*})}"'] & & h\mathrm{Fun}(X, A)^2 \arrow[dl, "{(\mathrm{cod}, \mathrm{dom})}"] \\
& h\mathrm{Fun}(X, A) \times h\mathrm{Fun}(X, A) \arrow[d, phantom, sloped, "\cong"] & \\
& h\mathrm{Fun}(X, A \times A) &
\end{tikzcd}$$

- We have a smothering functor over the cat $h\mathrm{Fun}(X, A \times A)$
- Surjectivity on objects ⤳ 1-cell induction, fullness ⤳ 2-cell induction
  conservativity ⤳ 2-cell conservativity ◁

- The functors $\ulcorner\alpha\urcorner : X \longrightarrow A^2$ are not unique

$$\begin{tikzcd} & \ulcorner\alpha\urcorner & \\ X \arrow[rr, bend left, ""{name=U, below}] \arrow[rr, bend right, ""{name=D, above}] & {} \arrow[u, squiggly] & A \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

- However, they are unique up to "fibered" isomorphisms:

Prop. Whiskering with $\begin{tikzcd} A^2 \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow k"] & A \end{tikzcd}$ induces

a bijection

$$\left\{ \begin{tikzcd} X \arrow[d, bend right=60, "t"'] \arrow[d, bend left=60, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\ A \end{tikzcd} \right\} \rightsquigarrow \left\{ \begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, "\ulcorner\alpha\urcorner"] & \\ A & & A \\ & A^2 \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd} \right\} \Big/ \cong$$

modulo the fibered isomorphisms

The fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, bend right=40, "\ulcorner\beta\urcorner"'] \arrow[dd, bend left=40, "\ulcorner\alpha\urcorner"] \arrow[dd, phantom, "\overset{\gamma}{\cong}"] & \\ A & & A \\ & A^2 \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}$$

s.t. $p_1\gamma = id_t$ & $p_0\gamma = id_s$

Proof: • The fibers of the smothering functor

$$\begin{tikzcd}
h\mathrm{Fun}(X, A^{2}) \arrow[rr] \arrow[dr, "{(p_{1*}, p_{0*})}"'] & & h\mathrm{Fun}(X, A)^{2} \arrow[dl, "{(\mathrm{cod}, \mathrm{dom})}"] \\
& h\mathrm{Fun}(X, A) \times h\mathrm{Fun}(X, A) &
\end{tikzcd}$$

are connected groupoids

- The objects of the fiber over $\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha"] & A \end{tikzcd}$ are functors

$X \longrightarrow A^{2}$, s.t. $\left(X \longrightarrow A^{2} \overset{k}{\Longrightarrow} k\right) = \alpha$ ← the generic arrow

whiskering

$$\begin{tikzcd}
X \arrow[d, bend right=50, "t"'] \arrow[d, bend left=50, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\
A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, dashed, "\ulcorner \alpha \urcorner"] \arrow[dd, bend right=80, "t"'] \arrow[dd, bend left=80, "s"] \\
A^{2} \arrow[d, bend right=50, "p_1"'] \arrow[d, bend left=50, "p_0"] \arrow[d, phantom, "\overset{k}{\Leftarrow}"] \\
A
\end{tikzcd}$$

- The morphisms — invertible 2-cells that whisker with $(p_1, p_0) : A^{2} \twoheadrightarrow A \times A$ to the identity 2-cell $(\mathrm{id}_t, \mathrm{id}_s)$

- The action of the smothering functor defines a bijection betw. the obj. - its codomain & their corresp. fibers

Uniqueness of arrow ∞-cats

Def. (fibered equivalence) A fibered equivalence over an ∞-cat B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

$$E \underset{B}{\simeq} F$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$

Prop. (uniqueness of arrow ∞-cats) $\forall$ isofibration $(e_1, e_0) : E \twoheadrightarrow A \times A$ fibered equivalent to $(p_1, p_0) : A^2 \twoheadrightarrow A \times A$ the 2-cell

$$\begin{tikzcd} E \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd}$$

encoded by the equivalence $e : E \xrightarrow{\sim} A^2$ satisfies the weak universal prop. And conversely...

The comma construction: in a 1-categorical setting

Consider a cospan $\mathcal{C} \xrightarrow{g} \mathcal{A} \xleftarrow{f} \mathcal{B}$ $\quad$ $\mathcal{A}, \mathcal{B}, \mathcal{C}$ – ordinary categories

Objects of $g\downarrow f$ are triples $(\underset{\mathcal{C}}{\underset{\cap}{C}}, \underset{\mathcal{B}}{\underset{\cap}{B}}, h)$, $h: g(C) \to f(B)$

a morphism in $\mathcal{A}$

Morphisms of $g\downarrow f$ are pairs (s_1, s_2), $s_1 : C \to C' \in \mathcal{C}_1$

$s_2 : B \to B' \in \mathcal{B}_1$

$$\begin{tikzcd}
g(C) \arrow[r, "g(s_1)"] \arrow[d, "h"'] & g(C') \arrow[d, "h'"] \\
f(B) \arrow[r, "f(s_2)"'] & f(B')
\end{tikzcd}$$

$$(s_1, s_2) \circ (s_1', s_2') = (s_1' \circ s_1, s_2' \circ s_2)$$

$$id_{(C,B,h)} = (id_C, id_B)$$

Equivalently,

$$\begin{tikzcd} g\downarrow f \arrow[r] \arrow[d, "{(p_0,p_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^2 \arrow[d, "{(p_0,p_1)}"] \\ \mathcal{C}\times\mathcal{B} \arrow[r, "g\times f"'] & \mathcal{A}\times\mathcal{A} \end{tikzcd}$$

$$\begin{array}{c} A^2 \\ \rotatebox[origin=c]{90}{$\in$} \\ \underbrace{(C, B, h : g(C) \to f(B))} \\ \rotatebox[origin=c]{-90}{$\in$} \\ g\downarrow f_0 \end{array}$$

(s_1, s_2, h) s.t.

$$(g\times f)(s_1,s_2) = (p_0,p_1)\underbrace{\left(\begin{tikzcd} g(C) \arrow[r, "h"] \arrow[d, "g(s_1)"] & f(B) \arrow[d, "f(s_2)"] \\ g(C') \arrow[r, "h'"] & f(B') \end{tikzcd}\right)}_{\substack{\rotatebox[origin=c]{-90}{$\in$} \\ (\mathcal{A}^2)_1}}$$

The comma construction in ∞-cosmos

Def. (comma ∞-cat) Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories in an ∞-cosmos $\mathcal{K}$. The comma ∞-cat $\mathrm{Hom}_A(f,g)$:

$$\begin{tikzcd}
f\downarrow g = \mathrm{Hom}_A(f,g) \arrow[r, "\ulcorner\varphi\urcorner"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, two heads, "{(p_1,p_0)}"] \\
C \times B \arrow[r, "g\times f"'] & A \times A
\end{tikzcd}$$

The top horizontal functor represents a 2-cell

$$\begin{tikzcd}
& f\downarrow g \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Leftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

the comma cone

in the homotopy 2-cat $\mathcal{K}$

By construction, the map $(p_1,p_0): f\downarrow g \twoheadrightarrow C\times B$ is an isofibration

Example. The ∞-cat of arrows is a special case of the comma construction

$A = A = A$ — the identity span

$$\begin{tikzcd}
\mathrm{id}\downarrow\mathrm{id} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d] \\
A\times A \arrow[r, equal] & A\times A
\end{tikzcd}$$

$\mathrm{id}\downarrow\mathrm{id} = A^2$

So, the generic arrow of the ∞-cat of arrows can be regarded as a comma cone:

$$\begin{tikzcd}
& A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
A \arrow[dr, equal] & \overset{\varphi}{\Longleftarrow} & A \arrow[dl, equal] \\
& A &
\end{tikzcd}$$

Pullbacks of isofibrations

- Pullbacks have weak 2-dim universal property

- It can be used to prove that equivalences pull back along isofibrations to equivalences

- In turn, it gives the equivalence invariance of pullbacks in an ∞-cosmos

Prop. (the weak universal property of the pullback)

$$\begin{tikzcd}
A \times E \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

(i) 1-cell induction: a commut. square over the cospan factors uniquely

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists ! x"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "a"'] & & \\
& A \times E \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

2-cell induction:

$$\begin{tikzcd}
X \arrow[rr, bend left, "gx"] \arrow[rr, bend right, "gx'"'] \arrow[rr, phantom, "\varepsilon\Downarrow"] \arrow[d, bend right=50, "qx'"'] \arrow[d, bend left=20, "qx"] \arrow[d, phantom, "\alpha\Leftarrow"] \arrow[drr, phantom, "="] & & E \arrow[d, "p"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[dr, bend left, "x"] \arrow[dr, bend right, "x'"'] \arrow[dr, phantom, "\exists\tau\Downarrow"] & & \\
& A \times E \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "p"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

Given functors $x, x' : X \longrightarrow A \times_B E$ & nat. transf. $\alpha : qx \Rightarrow qx'$, $\varepsilon : gx \Rightarrow gx'$ s.t. $p\varepsilon = f\alpha$, there exists a nat. transf. $\tau : x \Rightarrow x'$ s.t. $q\tau = \alpha$ & $g\tau = \varepsilon$

(iii) 2-cell conservativity: $\forall$

$$\begin{tikzcd}
X \arrow[r, bend left, "x"] \arrow[r, bend right] \arrow[r, phantom, "\tau\Downarrow"] & A \times_B E
\end{tikzcd}$$

if both $q\tau$ & $g\tau$ are iso then τ is an iso

Proof: • Consider the pullback diagram of quasi-cats

$$\begin{tikzcd}
\mathrm{Fun}(X, A \times_B E) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(X,E) \arrow[d, two heads, "p_*"] \\
\mathrm{Fun}(X,A) \arrow[r, "f_*"'] & \mathrm{Fun}(X,B)
\end{tikzcd}$$

• Apply the fact that the canonical functor

$$h(A \times_B E) \longrightarrow hA \times_{hB} hE$$

is smothering for any pullback diagram of quasi-cats of the form

$$\begin{tikzcd}
A \times_B E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

• So, apply it to the our case:

$$h\mathrm{Fun}(X, A \times_B E) \longrightarrow h\mathrm{Fun}(X,A) \times_{h\mathrm{Fun}(X,B)} h\mathrm{Fun}(X,E)$$

is smothering ◁

Prop. In any ∞-cosmos the pullback of an equivalence is an equivalence, i.e., ∞-cosmoi are right proper.

$$\begin{tikzcd}
F \arrow[r, "g", "\sim"'] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "\sim", "f"'] & B
\end{tikzcd}$$

Prop. (Pullback is an equiv. invariant construction in any ∞-cosmos)

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\wr"', "r"] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads] \arrow[d, "\wr"', "q"] \\
\bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"]
\end{tikzcd}$$

$C \times_A B \longrightarrow \bar{C} \times_{\bar{A}} \bar{B}$ is again an equivalence

Proof:

$$\begin{tikzcd}
C \arrow[r, dashed, "\sim"'] \arrow[rr, bend left, "g"] \arrow[d, "\wr"', "r"] & P \arrow[r, two heads] \arrow[d, "\wr"'] & A \arrow[d, "\wr"', "p"] \\
\bar{C} \arrow[r, "\sim"] \arrow[rr, bend right, "\bar{g}"'] & \bar{P} \arrow[r, two heads] & \bar{A}
\end{tikzcd}$$

2-of-3

the right properness

Brown's factorization used

Hence, $P \twoheadrightarrow A$ is equivalent to the map g

$$\begin{tikzcd}
C \times_A B \arrow[r, dashed, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \times_A B \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "f"] \\
C \arrow[r, "\sim"] & P \arrow[r, two heads] & A
\end{tikzcd}$$

- By right properness, the pullback of $P \twoheadrightarrow A$ along f is equiv. to the pullback of $g : C \to A$ along f and similarly for the lower maps
- So, it sufficies to consider

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "\wr"', "r"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "\wr"', "q"] \\
\bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"]
\end{tikzcd}$$

$$\begin{tikzcd}
& C \times_A B \arrow[rr, two heads] \arrow[dl] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dl, "f"] \arrow[dd, "q", "\wr"'] \\
C \arrow[rr, two heads, "g"'] \arrow[dd, "r"', "\wr"] \arrow[ddrr, phantom, "\lrcorner", very near start] & & A \arrow[dd, "p", "\wr"'] & \\
& \bar{C} \times_{\bar{A}} \bar{B} \arrow[rr, two heads] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \bar{B} \arrow[dl] \\
\bar{C} \arrow[rr, two heads, "\bar{g}"'] & & \bar{A} &
\end{tikzcd}$$

- They are pullback squares
- So, the back square is as well
- $C \times_A B \to \bar{C} \times_{\bar{A}} \bar{B}$ is the pullback of the equivalence q along an isofibration

◁

Prop. (maps between commas) A commutative diagram

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] & B \arrow[l, "f"'] \arrow[d, "q"] \\
\overline{C} \arrow[r, "\overline{g}"] & \overline{A} & \overline{B} \arrow[l, "\overline{f}"']
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\mathrm{Hom}_A(f,g) = f \downarrow g \arrow[r, dashed, "\mathrm{Hom}_p(q,r)"] \arrow[d, "(p_1{,}p_0)"'] & \overline{f} \downarrow \overline{g} = \mathrm{Hom}_{\overline{A}}(\overline{f},\overline{g}) \arrow[d, "(p_1{,}p_0)"] \\
C \times B \arrow[r, "r \times q"] & \overline{C} \times \overline{B}
\end{tikzcd}$$

If p, q and r are all isofibrations, all trivial fibrations, or all equivalences then the induced map is again an isofibration, trivial fibration, or equivalence, resp.

Proof:

$$\begin{tikzcd}
C \times B \arrow[r, "g \times f"] \arrow[d, "r \times q"'] & A \times A \arrow[d, "p \times p"'] & & A^{\mathbb{2}} \arrow[ll, two heads, "(p_1{,}p_0)"'] \arrow[d, "p^{\mathbb{2}}"] \arrow[dl, dashed] \\
& & \bullet \arrow[ul] \arrow[dr] \arrow[ul, phantom, "\lrcorner", very near start] & \\
\overline{C} \times \overline{B} \arrow[r, "\overline{g} \times \overline{f}"'] & \overline{A} \times \overline{A} & & \overline{A}^{\mathbb{2}} \arrow[ll, two heads, "(p_1{,}p_0)"]
\end{tikzcd}$$

the Leibniz tensor of $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{2}$ with p

Lemma.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[dd, "r"'] & A \arrow[dd, "p"] & & B \arrow[ll, two heads, "f"'] \arrow[dd, "q"] \arrow[dl, dashed, "z"'] \\
& & \bullet \arrow[ul] \arrow[dr] & \\
\bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & & \bar{B} \arrow[ll, two heads, "\bar{f}"]
\end{tikzcd}$$

in a cat of fibrant objects

If r and $z : B \longrightarrow A \times_{\bar{A}} \bar{B}$ are both triv. fib. or fib $\Rightarrow$ the induced pullback maps is again a triv. fib. or fib. resp.

$$\begin{tikzcd}
C \times_A B \arrow[rrr] \arrow[drr, two heads] \arrow[dr, dashed] \arrow[dd, dashed, "t"'] & & & B \arrow[drr, two heads] \arrow[dd, "q"'] \arrow[dr, "z"] & & \\
& \bullet \arrow[r, dashed] \arrow[rrr, dashed, bend left] \arrow[dl, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[rrr] \arrow[dd, "r"'] & & \bullet \arrow[r, dashed] \arrow[dl, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[dd, "p"] \\
\bar{C} \times_{\bar{A}} \bar{B} \arrow[rrr] \arrow[drr, two heads] & & & \bar{B} \arrow[drr, two heads] & & \\
& & \bar{C} \arrow[rrr] & & & \bar{A}
\end{tikzcd}$$

a pullback of z

a pullback of r

So, by the lemma, we are done in the case of fib. and triv. fib.

For equivalences, note that $r \times q$, $p \times p$ & p^2 are as well

$$\begin{tikzcd}
C \times B \arrow[r, "g \times f"] \arrow[d, "r \times q"'] & A \times A \arrow[d, "p \times p"'] & A^2 \arrow[l, two heads, "(p_1, p_0)"'] \arrow[d, "p^2"] \\
\bar{C} \times \bar{B} \arrow[r, "\bar{g} \times \bar{f}"'] & \bar{A} \times \bar{A} & \bar{A}^2 \arrow[l, two heads, "(p_1, p_0)"]
\end{tikzcd}$$

By the pullback invariance, we get an equivalence between the pullbacks $\triangleleft$

Prop. (the weak universal property of the comma ∞-category)

The comma cone

$$\begin{tikzcd}
& g\downarrow f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \arrow[ll, phantom, "\overset{\varphi}{\Leftarrow}"] \\
& A &
\end{tikzcd}$$

has a weak universal property in the homotopy 2-category given by 3 operations:

(i) 1-cell induction:

$$\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \arrow[ll, phantom, "\overset{\alpha}{\Leftarrow}"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] \arrow[d, dashed, "\ulcorner \alpha \urcorner"] & \\
& g\downarrow f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \arrow[ll, phantom, "\overset{\varphi}{\Leftarrow}"] \\
& A &
\end{tikzcd}$$

there exists a functor $\ulcorner \alpha \urcorner : X \to g\downarrow f$, s.t. $b = p_0 \ulcorner \alpha \urcorner$, $c = p_1 \ulcorner \alpha \urcorner$ & $\alpha = \varphi \ulcorner \alpha \urcorner$

(ii) 2-cell induction: Given $a, a' : X \longrightarrow \mathrm{Hom}_A(f,g)$ and nat. transf. τ_1 and τ_0 s.t.

given Such that given

$$\begin{tikzcd}
 & X & & & & & & X & \\
g\downarrow f & \overset{\tau_1}{\Longleftarrow} & g\downarrow f & & = & & g\downarrow f & \overset{\tau_0}{\Longleftarrow} & g\downarrow f \\
 & C & \overset{\varphi}{\Longleftarrow} & B & & C & \overset{\varphi}{\Longleftarrow} & B & \\
 & & A & & & & A & &
\arrow[from=1-2, to=2-1, "a'"']
\arrow[from=1-2, to=2-3, "a"]
\arrow[from=2-1, to=3-2, "p_1"']
\arrow[from=2-3, to=3-2, "p_1"]
\arrow[from=2-3, to=3-4, "p_0"]
\arrow[from=3-2, to=4-3, "g"']
\arrow[from=3-4, to=4-3, "f"]
\arrow[from=1-8, to=2-7, "a'"']
\arrow[from=1-8, to=2-9, "a"]
\arrow[from=2-7, to=3-6, "p_1"']
\arrow[from=2-7, to=3-8, "p_0"']
\arrow[from=2-9, to=3-8, "p_0"]
\arrow[from=3-6, to=4-7, "g"']
\arrow[from=3-8, to=4-7, "f"]
\end{tikzcd}$$

there exists a natural transformation $\tau : a \Rightarrow a'$ s.t.

$p_1\tau = \tau_1$ & $p_0\tau = \tau_0$, i.e.,

$$\begin{tikzcd}
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
g\downarrow f \arrow[dr, "p_1"'] & & g\downarrow f \arrow[ll, Rightarrow, shorten=3em, "\tau_1"'] \arrow[dl, "p_1"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, bend right=60, "a'"'] \arrow[d, bend left=60, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
g\downarrow f \arrow[d, "p_1"] \\
C
\end{tikzcd}$$

&

$$\begin{tikzcd}
 & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
g\downarrow f \arrow[dr, "p_0"'] & & g\downarrow f \arrow[ll, Rightarrow, shorten=3em, "\tau_0"'] \arrow[dl, "p_0"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, bend right=60, "a'"'] \arrow[d, bend left=60, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
g\downarrow f \arrow[d, "p_0"] \\
B
\end{tikzcd}$$

(iii) 2-cell conservativity: $\forall$ $\begin{tikzcd} X \arrow[r, bend left, "a"{name=U}] \arrow[r, bend right, "a'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\tau"] & g\downarrow f \end{tikzcd}$ if $p_1\tau$ & $p_0\tau$ are iso then τ is so

Proof: Apply the cosmological functor

$$Fun(X,-) : \mathcal{K} \to QCat$$

to

$$\begin{tikzcd} g\downarrow f \arrow[r, "\ulcorner\varphi\urcorner"] \arrow[d, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, "{(p_1,p_0)}"] \\ C\times B \arrow[r, "g\times f"'] & A\times A \end{tikzcd}$$

We will have

functors

$$\begin{tikzcd} Fun(X, Hom_A(g,f)) \cong Hom_{Fun(X,A)}(Fun(X,f), Fun(X,g)) \arrow[r, "\varphi"] \arrow[d, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fun(X,A)^2 \arrow[d, "{(p_1,p_0)}"] \\ Fun(X,C) \times Fun(X,B) \arrow[r, "Fun(X,g)\times Fun(X,f)"'] & Fun(X,A)\times Fun(X,A) \end{tikzcd}$$

Now, by the standard technique we get a smothering functor over $h\mathrm{Fun}(X, C\times B)$:

$$\begin{tikzcd}
h\mathrm{Fun}(X, \mathrm{Hom}_A(f,g)) \arrow[rr] \arrow[dr, "(p_{0*}, p_{1*})"'] & & \mathrm{Hom}_{h\mathrm{Fun}(X,A)}(h\mathrm{Fun}(X,f), h\mathrm{Fun}(X,g)) \arrow[dl, "(\mathrm{cod}, \mathrm{dom})"] \\
& h\mathrm{Fun}(X,C) \times h\mathrm{Fun}(X,B) &
\end{tikzcd}$$ ◁

The functors $\ulcorner\alpha\urcorner : X \to \mathrm{Hom}_A(f,g)$ induced by a fixed nat. transf.

$$\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

are unique up to fibered iso over $C \times B$

Prop. (1-cell induction is unique up to isomorphism)

$$\left\{\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}\right\} \xleftrightarrow[\text{with comma cone}]{\text{whiskering } \; 1-1} \left\{\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "\ulcorner\alpha\urcorner"] & \\ C & & B \\ & \underset{\substack{\parallel \\ g\downarrow f}}{\mathrm{Hom}_A(f,g)} \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}\right\} \Big/ \cong$$

↑ the fibered isomorphisms

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, bend right, "\ulcorner\beta\urcorner"'] \arrow[dd, bend left, "\ulcorner\alpha\urcorner"] \arrow[dd, phantom, "\overset{\gamma}{\cong}"] & \\ C & & B \\ & \underset{\substack{\parallel \\ g\downarrow f}}{\mathrm{Hom}_A(f,g)} \arrow[ul, "p_1"] \arrow[ur, "p_0"'] & \end{tikzcd}$$

$p_1\gamma = \mathrm{id}_c$

$p_0\gamma = \mathrm{id}_b$

$$\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] & f\downarrow g \end{tikzcd}$$

i.e., any two 1-cells $a, a' : X \to f\downarrow g$ over a weak comma object that are induced by the same comma cone $\alpha : fb \Rightarrow gc$ are isomorphic over $C \times B$

Proof: ① One way: fibres of smothering functors are connected groupoids

② Directly:

$$\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] & \overset{\alpha}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[d, dashed, "a"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\
& g\downarrow f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[d, dashed, "a'"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\
& g\downarrow f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

- We know that

$$\underbrace{p_0 a = p_0 a' \qquad\qquad p_1 a = p_1 a'} \qquad\qquad \underbrace{\varphi a = \varphi a'}$$

$$\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
g\downarrow f \arrow[dr, "p_1"'] & \overset{\mathrm{id}}{\Longleftarrow} & g\downarrow f \arrow[dl, "p_1"] \\
& C &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
g\downarrow f \arrow[dr, "p_0"'] & \overset{\mathrm{id}}{\Longleftarrow} & g\downarrow f \arrow[dl, "p_0"] \\
& C &
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] & & \\
g\downarrow f \arrow[dr, "p_1"'] & \overset{\mathrm{id}}{\Longleftarrow} & g\downarrow f \arrow[dl, "p_1"] \arrow[dr, "p_0"] & \\
& C \arrow[dr, "g"'] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl, "f"] \\
& & A &
\end{tikzcd}
=
\begin{tikzcd}
& & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\
& g\downarrow f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \overset{\mathrm{id}}{\Longleftarrow} & g\downarrow f \arrow[dl, "p_0"] \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl, "f"] & \\
& A & &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[d, "a'"] & \\
& g\downarrow f \arrow[dl] \arrow[dr] & \\
C \arrow[dr] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl] \\
& A &
\end{tikzcd}$$

$$\parallel$$

$$\begin{tikzcd}
& X \arrow[d, "a"] & \\
& g\downarrow f \arrow[dl] \arrow[dr] & \\
C \arrow[dr] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl] \\
& A &
\end{tikzcd}$$

- So, by 2-cell induction, $\exists \tau$

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\ g\downarrow f \arrow[dr, "p_0"'] & & g\downarrow f \arrow[ll, phantom, "\overset{id}{\Longleftarrow}"] \arrow[dl, "p_0"] \\ & B & \end{tikzcd} = \begin{tikzcd} X \arrow[d, bend right=60, "a'"'] \arrow[d, bend left=60, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ g\downarrow f \arrow[d, "p_0"] \\ B \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] & \\ g\downarrow f \arrow[dr, "p_1"'] & & g\downarrow f \arrow[ll, phantom, "\overset{id}{\Longleftarrow}"] \arrow[dl, "p_1"] \\ & C & \end{tikzcd} = \begin{tikzcd} X \arrow[d, bend right=60] \arrow[d, bend left=60] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ g\downarrow f \arrow[d, "p_1"] \\ C \end{tikzcd}$$

$p_0 \tau = id, \quad p_1 \tau = id$

- By 2-cell conservativity, $\tau : a' \Rightarrow a$ is iso

$$\begin{tikzcd} X \arrow[r, bend left=40, "a'"] \arrow[r, bend right=40, "a"'] \arrow[r, phantom, "\Downarrow \tau"] & g\downarrow f \end{tikzcd}$$

$\triangleleft$

As in the case of arrow categories

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] \arrow[dl, phantom, "\Downarrow\gamma"] & B \arrow[l, "f"'] \arrow[d, "q"] \arrow[dl, phantom, "\Downarrow\beta"] \\
\bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"]
\end{tikzcd}$$

induces a map between comma ∞-categories

functorial up to fibered isomorphism

$$\begin{tikzcd}
 & & \mathrm{Hom}_A(f,g) \arrow[dll, "p_1"'] \arrow[drr, "p_0"] & & \\
C \arrow[d, "r"'] \arrow[drr, "g"] & & \overset{\varphi}{\Leftarrow} & & B \arrow[d, "q"] \arrow[dll, "f"'] \\
\bar{C} \arrow[drr, "\bar{g}"'] & \Leftarrow \gamma & A \arrow[d, "p"] & \Leftarrow \beta & \bar{B} \arrow[dll, "\bar{f}"] \\
 & & \bar{A} & &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_A(f,g) \arrow[ddl, bend right, "rp_1"'] \arrow[ddr, bend left, "qp_0"] \arrow[d, dashed, "\beta\downarrow\gamma"] & \\
 & \mathrm{Hom}_{\bar{A}}(\bar{f},\bar{g}) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "\bar{g}"'] & \Leftarrow \varphi & B \arrow[dl, "\bar{f}"] \\
 & \bar{A} &
\end{tikzcd}$$

The mapping space

Comma ∞-cats can be used to define the internal mapping spaces

Def. The mapping space between two elements $x, y : 1 \longrightarrow A$ of an ∞-cat is $y \downarrow x = \mathrm{Hom}_A(x,y)$ – the comma ∞-cat

$$\begin{tikzcd}
\mathrm{Hom}_A(x,y) \arrow[r, "\ulcorner \varphi \urcorner"] \arrow[d, "{(p_1,p_0)}"'] & A^2 \arrow[d, "{(p_1,p_0)}"] \\
1 \arrow[r, "{(y,x)}"'] & A \times A
\end{tikzcd}$$

Prop. (mapping spaces are discrete)

For any pair of elements $x, y : 1 \longrightarrow A$ of an ∞-cat A, the mapping space $\mathrm{Hom}_A(x,y)$ is discrete

i.e., $\forall\, X \in \mathcal{K}$ $\mathrm{Fun}(X, \mathrm{Hom}_A(x,y))$ is a Kan complex

Proof: • We must show

$$h\mathrm{Fun}(X, \mathrm{Hom}_A(x,y)) \text{ is a groupoid } \forall X$$

•
$$\begin{tikzcd} X \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow"] & \mathrm{Hom}_A(x,y) \end{tikzcd}$$
is invertible, if
$$\begin{tikzcd} X \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\Leftarrow"] \\ \mathrm{Hom}_A(x,y) \arrow[d, "{(p_1, p_0)}"] \\ 1 \times 1 \end{tikzcd}$$
is an invertible 2-cell

— by 2-cell conservativity

• But this composite is id since 1 is 2-terminal ◁

Prop. (uniqueness of comma ∞-cats)

$\forall$ isofibration $(e_1, e_0) : E \twoheadrightarrow C \times B$ that is fibered equivalent to $\mathrm{Hom}_A(f,g) \twoheadrightarrow C \times B$ the 2-cell

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, two heads, "e_0"] & \\ C \arrow[dr, "g"'] & \overset{\varepsilon}{\Leftarrow} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

encoded by $e : E \xrightarrow{\sim} \mathrm{Hom}_A(f,g)$ satisfies the weak universal property of the comma ∞-cat

Conversely, if $(d_1, d_0) : D \twoheadrightarrow C \times B$ & $(e_1, e_0) : E \twoheadrightarrow C \times B$ are equipped with 2-cells

$$\begin{tikzcd} & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ C \arrow[dr, "g"'] & \overset{\delta}{\Leftarrow} & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \& \quad \begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\ C \arrow[dr, "g"'] & \overset{\varepsilon}{\Leftarrow} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

satisfying the weak univ. prop., then $D \underset{C \times B}{\cong} E$

Proof: $\boxed{\Rightarrow}$ Construct a smothering functor & enjoy

$\boxed{\Leftarrow}$

$$\begin{tikzcd}
 & D \arrow[dl, two heads, "d_1"'] \arrow[dr, two heads, "d_0"] & \\
C \arrow[dr, "g"'] & & B \arrow[ll, Rightarrow, "\delta"'] \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & D \arrow[d, dashed, "\ulcorner\delta\urcorner"] \arrow[ddl, two heads, bend right, "d_1"'] \arrow[ddr, two heads, bend left, "d_0"] & \\
 & E \arrow[dl, two heads] \arrow[dr, two heads] & \\
C \arrow[dr, "g"'] & & B \arrow[ll, Rightarrow, "\varepsilon"'] \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

&

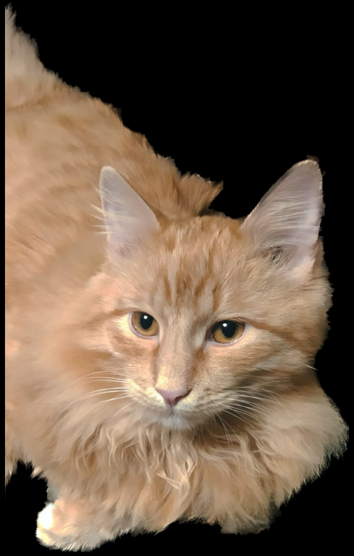
$$\begin{tikzcd}
 & E \arrow[dl, two heads, "e_1"'] \arrow[dr, two heads, "e_0"] & \\
C \arrow[dr, "g"'] & & B \arrow[ll, Rightarrow, "\varepsilon"'] \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & E \arrow[d, dashed, "\ulcorner\varepsilon\urcorner"] \arrow[ddl, two heads, bend right, "e_1"'] \arrow[ddr, two heads, bend left, "e_0"] & \\
 & D \arrow[dl, two heads] \arrow[dr, two heads] & \\
C \arrow[dr, "g"'] & & B \arrow[ll, Rightarrow, "\delta"'] \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

uniqueness of 1-cell induction (up to iso)

$D \cong E$

$\Uparrow$

Notice $\varepsilon\ulcorner\delta\urcorner\ulcorner\varepsilon\urcorner = \varepsilon$ & $\delta\ulcorner\varepsilon\urcorner\ulcorner\delta\urcorner = \delta$ $\Longrightarrow$ $\ulcorner\delta\urcorner\ulcorner\varepsilon\urcorner \underset{C\times B}{\cong} id_E$ & $\ulcorner\varepsilon\urcorner\ulcorner\delta\urcorner \underset{C\times B}{\cong} id_D$ $\lhd$



Merci

beaucoup!