§1. Products & coproducts in BornCoarse

- X $\quad \mathcal{B}_{min} := \mathcal{B}\langle\emptyset\rangle$ — all finite subsets of X

It is compatible with $\mathcal{C}_{min} := \mathcal{C}\langle\emptyset\rangle$

finite subsets of $diag_X$

$B[U] = \{x \in X \mid \exists b \in B \ (x,b) \in U\}$

- $\mathcal{B}_{max} := P(X)$

$\mathcal{C}_{max} := P(X \times X)$

$\mathcal{B}_{max}$ is comp. with all coarse structures

$\Rightarrow \mathcal{B}_{max}$ is comp. with $\mathcal{C}_{max}$

Notation: $\quad X_{min, min}, \quad X_{max, max}$

Lemma. $(-)_{min, max} \quad (-)_{max, min} : \mathrm{Set} \to \mathrm{BornCoarse}$

$$(\underbrace{-)_{min}}_{\text{the coarse structure}}, \underbrace{{}_{max}}_{\text{the bornology}} : \mathrm{Set} \rightleftarrows \mathrm{BornCoarse} : \underbrace{U}_{\text{the forgetful functor}}$$

▷ $A \in Set \quad X \in BornCoarse$

$$\{A_{min,max} \longrightarrow X\} \longleftrightarrow \{A \longrightarrow UX\}$$ ◁

Prop. The category BornCoarse does not have a final object

▷ $U : BornCoarse \longrightarrow Set$ – right adj to $(-)_{min,max}$

$\Rightarrow$ it preserves limits

Hence, if it was the case $\Rightarrow$ the underlying set would be $*$

[lim $\emptyset$ = a final object]

Assume that $*$ is final $\Rightarrow$ $X \longrightarrow * \Rightarrow X \in \mathcal{B}$ ◁

Remark. The final obj. is the only non-existing limit in BornCoarse

What about colimits?

Assume that all finite sets are bounded

Example.

$$\begin{tikzcd} \mathbb{N}_{min,max} \arrow[r, "id"] \arrow[d, "id"'] & \mathbb{N}_{max,max} \\ \mathbb{N}_{min,min} & \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{N}_{min,max} \arrow[r, "id"] \arrow[d, "id"'] & \mathbb{N}_{max,ma} \arrow[d, "\alpha"] \\
\mathbb{N}_{min,min} \arrow[r, "\beta"'] & T
\end{tikzcd}$$

just commutes

there is no comm. diagrams like this

$t := \alpha(0)$ Then $B := \{t\}$ is bounded

$\mathbb{N}\times\mathbb{N}$ is controlled in $\mathbb{N}_{max,max}$ (i.e., $\mathbb{N}\times\mathbb{N} \in \mathcal{E}(\mathbb{N}_{max})$)

$\Rightarrow \mho := (\alpha\times\alpha)(\mathbb{N}\times\mathbb{N}) \in \mathcal{E}_T$

$\Rightarrow \mho[B] \in \mathcal{B}_T$

By properness of β

$$\beta^{-1}(\mho[B]) \in \mathcal{B}(\mathbb{N}_{min,min}) \Rightarrow$$

$\Rightarrow \beta^{-1}(\mho[B])$ is finite

But from the commut. of the diagram

$$\beta^{-1}(\mho[B]) = \mathbb{N},$$

since $\forall n \in \mathbb{N}$

$$\beta(n) = \alpha(n) \in \mho[B] = \mho[\{\alpha(0)\}]$$

$(\alpha(n), \alpha(0)) \in \mho$

$\widetilde{\text{BornCoarse}}$ — the cat of generalized born. coarse spaces

[see the article Daniel Heiss '19]

Lemma. BornCoarse has all coproducts

$\triangleright$ $(X_i, \mathcal{C}_i, \mathcal{B}_i)_{i\in I}$ — a family of born. coarse spaces

Define $(X, \mathcal{C}, \mathcal{B})$ by

$$X := \coprod_{i\in I} X_i \qquad \mathcal{C} := \mathcal{C}\langle \bigcup_{i\in I} \mathcal{C}_i \rangle$$

$$\mathcal{B} := \{ B \subseteq X \mid \forall i\in I : B\cap X_i \in \mathcal{B}_i \}$$

$\triangleleft$

$X_i \hookrightarrow X$ are morphisms

Lemma. BornCoarse has all non-empty products

$\triangleright$ $(X_i, \mathcal{C}_i, \mathcal{B}_i)_{i\in I}$

$$X := \prod_{i\in I} X_i$$

$$\mathcal{C} := \mathcal{C}\langle \prod_{i\in I} U_i \rangle \quad \forall (U_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}_i$$

$$\mathcal{B} := \mathcal{B}\langle B_j \times \prod_{i\in I\setminus\{j\}} X_i \mid j\in I,\ B_j \in \mathcal{B}_j \rangle$$

And the projections from $(X, \mathcal{C}, \mathcal{B})$ to the factors are morphisms $\triangleleft$

Def. $(X_i, \mathcal{C}_i, \mathcal{B}_i) \rightsquigarrow \coprod_{i \in I}^{free} (X_i, \mathcal{C}_i, \mathcal{B}_i)$ — the free union

1. The underl. set is $\coprod_{i \in I} X_i$

2. $\mathcal{C} := \mathcal{C}\langle \bigcup_{i \in I} U_i \rangle \quad \forall (U_i)_{i \in I} \quad U_i \in \mathcal{C}_i$

3. $\mathcal{B} = \mathcal{B}\langle \bigcup_{i \in I} \mathcal{B}_i \rangle$

Remark. The free union $\neq$ the coprodt

The free union plays a role of additivity of coarse homology theories

Def. $\coprod_{i \in I}^{mixed} (X_i, \mathcal{C}_i, \mathcal{B}_i)$

$\mathcal{C} := \mathcal{C}^{coproduct} = \mathcal{C}\langle \bigcup \mathcal{C}_i \rangle$

$\mathcal{B} := \mathcal{B}^{free} = \mathcal{B}\langle \bigcup_{i \in I} \mathcal{B}_i \rangle$

We have the morphisms

$$\coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I}^{\text{mixed}} X_i \longrightarrow \coprod_{i\in I}^{\text{free}} X_i$$

$\pi_0^{\text{coarse}}(X,\mathcal{C})$ — coarse components of $(X,\mathcal{C})$

$$R_{\mathcal{C}} := \bigcup_{U\in\mathcal{C}} U \subseteq X\times X \text{ — an equivalence relation}$$

$(X,\mathcal{C})$ is coarsely connected if $\#\pi_0^{\text{coarse}}(X,\mathcal{C}) = 1$

$$\pi_0^{\text{coarse}}\Big(\coprod_{i\in I} X_i\Big) \cong \coprod_{i\in I} \pi_0^{\text{coarse}}(X_i)$$

§2. τ_X on BornCoarse & the sheaves

Def. $\mathcal{Y}$ is called a big family on X

$\mathcal{Y} = (Y_i)_{i\in I}$ — a filtered family of subs. of X

s.t. $\quad U[Y_i] \subseteq Y_j \quad \forall i\in I\ \forall U\in\mathcal{C}\ \exists j\in I$

Example. $(X, \mathcal{C})$

$A \subseteq X \rightsquigarrow$ one can form

$$\{A\} := (U[A])_{U \in \mathcal{C}}$$

Example. $(X, \mathcal{C}, \mathcal{B})$

$\mathcal{B}$ is a big family

Def. $(\mathcal{Z}, \mathcal{Y})$ — a complementary pair

$\mathcal{Z} \subseteq X$, $\mathcal{Y}$ a big family

s.t. $\exists i \in I \quad \mathcal{Z} \cup Y_i = X$

$$Spc^{la} \simeq sSet^{la}[W^{-1}]$$

Def. τ_χ on BornCoarse s.t. τ_χ-sheaves are exactly the presheaves satisfying descent for compl. pairs

Def. (The descent condition)

$E \in PSh(\text{BornCoarse})$

$E(\emptyset) \simeq *$ & $\forall (\mathcal{Z}, \mathcal{Y})$ on X

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(\mathcal{Z}) \arrow[d] \\ E(Y) \arrow[r] & E(\mathcal{Z} \cap Y) \end{tikzcd}$$ is cartesian

Lemma. τ_{χ} is subcanonical

§3. Coarse Equivalences

X, X' — born. coarse spaces

$f_0, f_1 : X \to X'$ — a pairs of morph.

Def. f_0 & f_1 are said to be close to each other if $(f_0 \times f_1)(\mathrm{diag}_X)$ is entourage of X'

Def. $f : X \to X'$ is an equiv. if $\exists g : X' \to X$ s.t. $f \circ g$ & $g \circ f$ are close to the respective identities

Example. $Y \subset X$, $U \in \mathcal{C}$, $U \supset \mathrm{diag}_X$

Then $Y \hookrightarrow U[Y]$ is an equiv. in BornCoarse

The inverse map $g: \mho[Y] \to Y$
$$\begin{array}{c} \mho[Y] \\ \cup \\ x \end{array}$$
$$\exists g(x): \ (x, g(x)) \in \mho$$

Example. (X, d) & (X', d') — metric spaces

$f: X \to X'$ — a quasi-isometry if $\exists C, D, E \in (0, \infty)$, s.t.

$$C^{-1} d'(f(x), f(y)) - D \leq d(x, y) \leq C d'(f(x), f(y)) + D$$

& $\forall x' \in X' \ \exists x \in X$ s.t. $d'(f(x), x') \leq E$

Prop. If f is an quasi-isometry $\Rightarrow f: X_d \to X'_{d'}$ is an equivalence in BornCoarse

$\triangleright$ As for $g: X'_{d'} \to X_d$ take a map

$$\begin{array}{ccc} x' & \longmapsto & x \\ \cap & & \cap \\ X'_{d'} & & X_d \end{array} \quad \text{s.t. } d'(f(x), x') \leq E$$

$$gf: X_d \to X_d$$

$$(gf \times id_{X_d})(diag_{X_d}) = \{(\tilde{x}, x)\}$$

$$d'(f(\tilde{x}), f(x)) \leq E \Rightarrow$$

$$\Rightarrow d(\tilde{x}, x) \leq C d'(f(\tilde{x}), f(x)) + D \leq E + D$$

$$(fg \times id_{X'_d})(diag_{X_d}) = \{(f(x), x')\}$$

$x' \xmapsto{g} x$ s.t. $d'(f(x), x') \leq E$ $\triangleleft$

Example. Some invariants for X:

1. The bornology of X is countably generated
2. The number of generators of $\mathcal{E}$
3. $\pi_0^{coarse}(-)$ — a Set-valued invariant w.r.t. to equivalences

We want: $E \in Sh(BornCoarse)$

$$E : Equiv.(BornCoarse) \longrightarrow W\mathcal{E}(Sp^{la})$$

Introduce "a segment" $\{0,1\}_{max, max}$

Prop. $f_0, f_1 : X \to X'$ are close to each other

$\iff$ one can combine them to a single map

$$\{0,1\} \otimes X \longrightarrow X'$$

$\forall X \in BornCoarse$, $\{0,1\} \otimes X \to X$ this arrow is morphism (a projection)

$\triangleright$ $(f_0 \times f_1)(diag_X)$ — an entourage in X'

$$f_0 \times f_1 \rightsquigarrow h : \{0,1\} \otimes X \to X'$$

$$h(0,x) = f_0 \qquad (h(0,x), h(1,x)) \in \mathcal{E}_X$$

$$h(1,x) = f_1$$

$\triangleleft$

Def. $E \in Sh(\mathrm{BornCoarse})$ — coarsely invariant if $\forall X \in \mathrm{BornCoarse}$ $\{0,1\} \otimes X \to X$ induces

$$E(X) \longrightarrow E(\{0,1\} \otimes X)$$

Lemma. The coarsely invariant sheaves $Sh^{\{0,1\}}(\mathrm{BornCoarse})$ form a full localizing subcategory of $Sh(\mathrm{BornCoarse})$

$\triangleright$ Recall

Def. • $\exists\, \mathcal{C}$ — an ∞-cat, S — a collect of morphisms

$Z \in \mathcal{C}$ is called S-local if

$\forall (s : X \to Y) \in S$

the comp with s

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, Z) \text{ — iso}$$

in homotop. cats of spaces

• $(f : X \to Y) \in \mathcal{C}$ is called an S-equiv

if $\forall$ S-local obj. Z

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\text{of}} \mathrm{Map}_{\mathcal{C}}(X, Z) \text{ — iso}$$

By the Yoneda lemma $\Rightarrow$ $Sh^{\{0,1\}}(BornCoarse)$ – local w.r. to the

$$\mathcal{Y}'(\{0,1\}\otimes X) \longrightarrow \mathcal{Y}'(X) \quad \forall X \in BornCoarse$$

Prop. (Lurie, HTT, 5.5.4.15)

$\mathcal{C}$ – a presentable ∞-cat & S – a small collection of morph. of $\mathcal{C}$

$\mathcal{C}' \subset \mathcal{C}$ – S-local objects

Then • $\iota : \mathcal{C}' \subseteq \mathcal{C}$ has a left adj $\mathcal{L}$

• $f \in \mathcal{C}$ is an S-equiv. $\Leftrightarrow$ $\mathcal{L}f$ is an equiv.

$$\mathcal{L} =: H^{\{0,1\}} : Sh(BornCoarse) \rightleftarrows Sh^{\{0,1\}}(BornCoarse) : \iota$$