X — a set

$\mathcal{P}(X)$ — the set of all subsets of X

Def. A bornology on X is a subset

$$\mathcal{B} \subseteq \mathcal{P}(X)$$

1. $\bigcup_{B \in \mathcal{B}} B = X$
2. $A \subseteq \underset{\mathcal{B}}{\underset{\cap}{B}} \Rightarrow A \in \mathcal{B}$
3. $B_1, \dots, B_n \in \mathcal{B} \Rightarrow B_1 \cup \dots \cup B_1 \in \mathcal{B}$

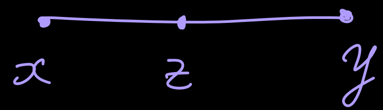
$\mathcal{B}$ — bounded subsets

Def. D is locally finite (countable) if $\forall B \in \mathcal{B}$ $B \cap D$ is finite (countable)

$$U \subseteq X \times X$$

$$U^{-1} := \{(x,y) \in X \times X \mid (y,x) \in U\}$$

$$U \circ U' := \{(x,y) \in X \times X \mid \exists z \in X \ (x,z) \in U \ \& \ (z,y) \in U'\}$$



Def A coarse structure on X is a subset $\mathcal{E}$ of $\mathcal{P}(X \times X)$ which is closed under $-\circ-$, $\underset{\text{finite}}{\cup}$, $-^{-1}$, $A \subseteq B$ & $\mathcal{E}$ should contain $\operatorname{diag}(X)$

1. $A_1 \cup \ldots \cup A_n \in \mathcal{E}$, $A_i \in \mathcal{E}$
2. $A_i \circ A_j \in \mathcal{E} \quad \forall A_i, A_j$
3. $A^{-1} \in \mathcal{E}$, if $A \in \mathcal{E}$
4. $A_2 \subseteq A_1$ & $A_1 \in \mathcal{E} \Rightarrow A_2 \in \mathcal{E}$

$\exists\, U \subseteq X \times X$, $B \subseteq X$

$$U[B] := \{ x \in X \mid \exists\, b \in B : (x, b) \in U \}$$

it is called the U-thickening of B

The elements of $\mathcal{E}$ — entourage of X or controlled subsets

Coarse structure on X = some bornology on $X \times X$ + groupoidal-like properties

$(X, \mathcal{E}, \mathcal{B})$ – a bornological coarse space

if a coarse structure $\mathcal{E}$ & a bornology are compatible:
if $\forall$ controlled thickening of a bounded subset is again bounded

Morphisms

f is proper if $\forall B' \in \mathcal{B}'$
$$f^{-1}(B') \in \mathcal{B}$$

f – bornological if $\forall B \in \mathcal{B}$
$$f(B) \in \mathcal{B}'$$

$$f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$$

f is controlled if $\forall U \in \mathcal{E}$
$$(f \times f)(U) \in \mathcal{E}'$$

Def. $f : (X, \mathcal{E}, \mathcal{B}) \longrightarrow (X', \mathcal{E}', \mathcal{B}')$ is a map
if it is a map between X and X' s.t. it is

proper & controlled

So, bornological coarse spaces form a small cat
Born Coarse

Examples

1. X — a set

$A \in \mathcal{P}(X \times X)$

$\mathcal{C}\langle A \rangle$ — a minimal coarse structure

$\mathcal{W} \in \mathcal{P}(X) \rightsquigarrow \mathcal{B}\langle \mathcal{W} \rangle$

$U \in A$ — any entourage

$B \in \mathcal{W}$ — a generating bounded subset

$U[B]$ & $U^{-1}[B]$ are contained in finite unions of members of $\mathcal{W} \Rightarrow \mathcal{B}\langle \mathcal{W} \rangle$ & $\mathcal{C}\langle A \rangle$ are compatible

② (Discrete bornological coarse spaces)

X — a set

$$\mathcal{E}_{min} := \mathcal{E}\langle \emptyset \rangle$$

it is generated by the empty set

$\mathcal{E}_{min}$ has diag(X) & all its subsets

$\forall$ bornological structure $\mathcal{B}$,

$$\mathcal{B} \ \& \ \mathcal{E}_{min}\langle \emptyset \rangle$$

are compatible

$$U[B] = \{ b \in B \mid \exists b \in B \ (b, b) \in U \}$$

In particular, take $\mathcal{B} = \mathcal{B}_{min}$

all finite subsets of X

③ $(X, \mathcal{E})$ — a coarse space

$\exists$ a min compatible bornology $\mathcal{B}$

it consists of the subsets of X which are bounded for some entourage of X

"$\mathcal{B}$ is generated by $\mathcal{E}$"

$V[B]$ — it is bounded $\forall V \in \mathcal{E}$

$$B \times B \subseteq V[B]$$

4 $(X', \mathcal{C}', \mathcal{B}')$ – a bornological coarse space

$f: X \to X'$ – a map of sets

$$f^*\mathcal{C} := \mathcal{C}\langle\{(f\times f)^{-1}(U') \mid U' \in \mathcal{C}'\}\rangle$$

$$f^*\mathcal{B} := \mathcal{B}\langle\{f^{-1}(B') \mid B' \in \mathcal{B}'\}\rangle$$

Then
$$f: (X, f^*\mathcal{C}', f^*\mathcal{B}') \to (X', \mathcal{C}', \mathcal{B}')$$
is a morphism in BornCoarse

5 (X, d) – a metric space

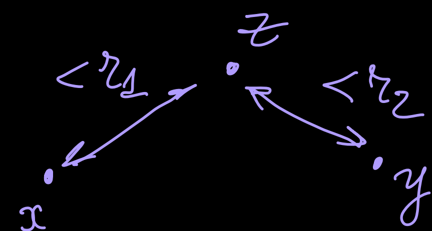
a metric

$$\mathcal{B}_d := \mathcal{B}\left\langle\left\{B_d(x, r) \;\middle|\; \begin{array}{c} r \geq 0 \\ x \in X \end{array}\right\}\right\rangle$$

$$U_r := \{(x, y) \in X \times X \mid d(x, y) < r\}$$

$$U_{r_1} \circ U_{r_2} = U_{r_1 + r_2}$$

$$U_{r_1}^{-1} = U_{r_1}$$



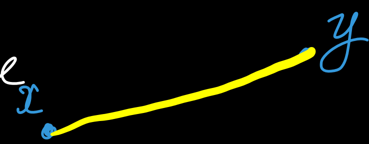
$$U_1 \cup \dots \cup U_{r_k} = U_{\max\limits_{i=1,\dots,k} r_i}$$

$$U \subset U_R \in \mathcal{C} \Rightarrow \underbrace{U \in \mathcal{C}}_{\text{we require}}$$

$\mathcal{C}_d := \mathcal{C}\langle\{U_r \mid r\in(0,\infty)\}\rangle$ — the coarse structure

Lemma. (X,d) — a path metric space

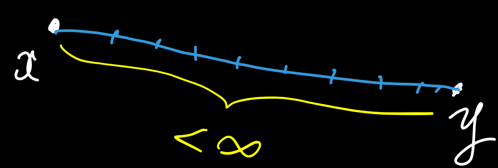
$\mathcal{C}_d$ — the associated coarse structure



Then $\exists$ an entourage U in $\mathcal{C}_d$ s.t.

$$\mathcal{C}_d = \mathcal{C}\langle U\rangle$$

$\triangleright$



$(x,y) = (x,x_1)\to\ldots\to(x_n,y)$ $\triangleleft$

6) Γ — group

$\mathcal{B}_{min}$ — the minimal born. struct. on Γ

$\Gamma(B\times B)$ — the Γ-invariant entourage

$B\in\mathcal{B}_{min}$

$\mathcal{C}_{can} := \mathcal{C}\langle\Gamma(B\times B) \mid B\in\mathcal{B}_{min}\rangle$

— the canonical coarse structure

Lemma. BornCoarse has all non-empty products

Lemma. BornCoarse has all coproducts

$(X, \mathcal{C}, \mathcal{B})$

U — an entourage of X

$$X_U := (X, \mathcal{C}\langle U \rangle, \mathcal{B})$$

Prop. $X \cong \operatorname{colim}_{U \in \mathcal{C}} X_U$

Example $(X, \mathcal{C}, \mathcal{B})$

$(X', \mathcal{C}', \mathcal{B}')$

$$(X, \mathcal{C}, \mathcal{B}) \otimes (X', \mathcal{C}', \mathcal{B}') =$$

$$= (X \times X', \mathcal{C} \times \mathcal{C}', \mathcal{B} \times \mathcal{B}')$$

$\otimes$ gives a sym. monoidal struct. on BornCoarse

$$\mathrm{PSh}(\mathrm{BornCoarse}) := \mathrm{Fun}(\mathrm{BornCoarse}^{op}, \underset{\sim}{\mathrm{Sp}}^{(\mathrm{la})})$$

∞-cats

Descent

$\mathcal{T}_X$

Def. $\mathcal{Y}$ is a big family on X

$\mathcal{Y} = \{Y_i\}_{i\in I}$ $\quad Y_i \subset X$, s.t. $\forall i \in I$ $\forall U$ $\exists j \in I$ $U[Y_i] \subseteq Y_j$

Example. $(X, \mathcal{C}, \mathcal{B})$

In this case, $\mathcal{B}$ is a big family on X

Def. $(Z, \mathcal{Y})$ s.t. $\exists i$ $Z \cup Y_i = X$

Example $f : X' \to X$

$(Z, \mathcal{Y})$

One can pull it back along f: $(f^{-1}(Z), f^{-1}(\mathcal{Y}))$

Def. $PSh(\mathcal{C}) \simeq \mathrm{Fun}^{\lim}(PSh(\mathcal{C})^{op}, Spc^{la})$

$$E(F) = \lim_{(\mathcal{L}'(X) \to F) \in \mathcal{C}/_F} E(X)$$

$$E(\mathcal{L}(X)) = E(X)$$

$$\mathcal{Y} = (Y_i)_i \rightsquigarrow \mathcal{L}'(\mathcal{Y}) = \operatorname*{colim}_{i \in I} \mathcal{L}'(Y_i)$$

$$\uparrow$$

$$PSh(\mathrm{BornCoarse})$$

Def. E satisfies descent for compl. pairs if

$E(\emptyset) \simeq *$

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(Z) \arrow[d] \\ E(Y) \arrow[r] & E(Z \cap Y) \end{tikzcd}$$

in Spc

Lemma. The Grothendieck top. τ_χ s.t. all τ_χ-sheaves are exactly the presheaves which satisfy the descent for compl. pairs

Lemma. The Grothendieck top. $\mathcal{T}_\chi$ on BornCoarse

Subcanonical

$\triangleright$ $X' \in$ BornCoarse $\quad \mathcal{L}(X')$??? – is a sheaf

$\mathcal{L}(X')$

$$\iota : Set^{la} \longrightarrow sSet^{la} \longrightarrow Sp^{la}$$

$$y : BornCoarse \longrightarrow PSh_{Set^{la}}(BornCoarse)$$

$$\mathcal{L}(X') = \iota \circ y\,(X')$$

$y(X')$

(Z, Y) – a compl. pair

$$y(X')(X) \longrightarrow y(X')(Z) \times_{y(X')(Z \cap Y)} y(X')(Y)$$

$$Z \cup Y_{i_0} = X$$

$f : \mathcal{Z} \longrightarrow X'$ $\quad g_i : Y_i \longrightarrow X'$

s.t. $\mathcal{Z} \cap Y_i$

$h : X \longrightarrow X'$

4