**Def.** Nerve Realization Context is a functor

$F : \mathcal{C} \longrightarrow \mathcal{D}$ ($\mathcal{C}$ small, $\mathcal{D}$ locally small)

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\mathcal{L}'"'] & \arrow[d, phantom, "\eta \Downarrow"] & \mathcal{D} \\ & {[\mathcal{C}^{op}, Set]} \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$

**Theorem** (Univ. property of $[\mathcal{C}^{op}, Set]$)

1) There is $\mathrm{Lan}_{\mathcal{L}'_{\mathcal{C}}} F$. $\mathrm{Lan}_{\mathcal{L}'_{\mathcal{C}}} \dashv - \mathcal{L}'_{\mathcal{C}}$,
the unit of this adjunction is invertible

2) The ess. image of $\mathrm{Lan}_{\mathcal{L}'}$ consists of those $\bar{F} : [\mathcal{C}^{op}, Set] \longrightarrow \mathcal{D}$ that preserves all colimits

3) If $\mathcal{D} = [\mathcal{E}^{op}, Set]$, this ess. image is equivalent to the subcat of left adjoints $F : [\mathcal{C}^{op}, Set] \to [\mathcal{E}^{op}, Set]$

**Proof.** 1) • $\forall\, F : \mathcal{C} \to \mathcal{D}$ extends uniq. to $\bar{F} : [\mathcal{C}^{op}, Set] \to \mathcal{D}$ (up to isomorphism)

• $\exists$ an adjunction $\mathrm{Lan}_{\mathcal{L}'_{\mathcal{C}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} - \circ \mathcal{L}'_{\mathcal{C}}$
that has invertible unit $\eta$

$\bar{F} \Rightarrow (F \longmapsto \bar{F})$ is a functor

$\forall\, G : [\mathcal{C}^{op}, Set] \longrightarrow \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\mathcal{L}'"] \arrow[dr, "\mathcal{L}'"'] & \arrow[d, phantom, "\eta \Downarrow"] & \mathcal{D} \\ & {[\mathcal{C}^{op}, Set]} \arrow[ur, dashed, ""{name=A}] \arrow[ur, bend right=60, "G"'{name=B}] \arrow[from=A, to=B, phantom, "\overset{\varepsilon}{\Rightarrow}"] & \end{tikzcd}$$

$\exists!\ \varepsilon : \mathrm{Lan}_{\mathcal{L}'}(G\mathcal{L}') \Rightarrow G$ is the counit of our adj

$\eta : \mathrm{Lan}_{\mathcal{L}'} G \circ \mathcal{L}' \overset{?}{\cong} G$

**Lemma.** $\mathrm{Lan}_K F \circ K \cong F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"'] & \end{tikzcd}$$

$K$ fully faithful

$\triangleright$ $\mathrm{Lan}_K F(d) = \int^{c\in\mathcal{C}_0} \mathcal{D}(Kc, d) \cdot Fc$ copower or tensor prod

$(\mathrm{Lan}_K F)(K(c)) = \int^{x\in\mathcal{C}_0} \mathcal{D}(Kx, Kc) \cdot Fx \cong$

$\cong \int^{x\in\mathcal{C}_0} \mathcal{C}(x,c) \cdot Fx \overset{\mathcal{L}}{\cong} Fc$ $\triangleleft$

2) Suppose that $G : [\mathcal{C}^{op}, \mathrm{Set}] \to \mathcal{D}$ is continuous $\overset{?}{\Rightarrow}$

$\overset{?}{\Rightarrow} (\varepsilon_G : \mathrm{Lan}_{\mathcal{L}'_{\mathcal{C}}}(G \circ \mathcal{L}'_{\mathcal{C}}) \Rightarrow G)$ is invertible

$\mathrm{Lan}_{\mathcal{L}'}(G\circ\mathcal{L}')(P) \cong \int^{c\in\mathcal{C}} [\mathcal{C}^{op}, \mathrm{Set}](\mathcal{L}'c, P) \cdot G\mathcal{L}'c \cong$

$\cong \int^{c\in\mathcal{C}} Pc \cdot G\mathcal{L}'c \xrightarrow{\sim} G\left(\int^{c\in\mathcal{C}} Pc \cdot \mathcal{L}'c\right) \cong GP$

Vice Versa, if we have $G \cong \mathrm{Lan}_{\mathcal{L}'}(G\mathcal{L}')$ then

$G(\operatorname{colim}_J P_j) \cong \int^{c\in\mathcal{C}} \operatorname{colim}_J((P_j)c) \cdot G(\hom(-,c)) \cong$

$P : J \to [\mathcal{C}^{op}, \mathrm{Set}]$

Fubini theorem

$\cong \operatorname{colim}_J \underbrace{\int^{c\in\mathcal{C}} (P_j)c \cdot G(\hom(-,c))}_{(\mathrm{Lan}_{\mathcal{L}'}(G\mathcal{L}'))(P_j)} \cong$

$\cong \operatorname{colim}_{J} G(P_j)$

3) $\mathcal{L} : [\mathcal{E}^{op}, Set] \longrightarrow [\mathcal{E}^{op}, Set]$ — continuous functor

Then it has a right adj given by

$$\int^{c\in\mathcal{E}} [\mathcal{E}^{op}, Set](\mathcal{L}c, X)\cdot c \qquad \triangleleft$$

Example

$$\begin{tikzcd} \Delta \arrow[rr, "\Delta^{\bullet}"] \arrow[dr, "\mathcal{L}'"'] & & \mathcal{E} \ / \text{ cocomplete} \\ & sSet \arrow[ur, dashed, "\mathcal{L}"'] & \end{tikzcd}$$

$$\mathcal{L}X := \int^{n\in\Delta} sSet(\Delta^n, X)\cdot\Delta^n \cong \int^{n\in\Delta} X_n\cdot\Delta^n \cong \operatorname{coeq}\left[\coprod_{[n]\to[m]} X_m\cdot\Delta^n \rightrightarrows \coprod_{[n]\in\Delta} X_n\right]$$

We know that $\mathcal{L}\Delta^n \cong \Delta^n$

$\mathcal{L}$ is a colimit $\Rightarrow$ $\mathcal{L}$ commutes with colimits $\Rightarrow$ $\mathcal{L}$ has a right adj. $R : \mathcal{E} \to sSet$

$$(Re)_n \cong sSet(\Delta^n, Re) \cong \mathcal{E}(\mathcal{L}\Delta^n, e) \cong \mathcal{E}(\Delta^n, e)$$

$\Rightarrow$ $n$-simpl. of $Re$ are maps in $\mathcal{E}$ from $\Delta^n$ to $e$

Example 2. $\Delta \longrightarrow Top$, s.t. $\Delta^n$ is the stand. top. sim

$|-| : sSet \rightleftarrows Top : S$

$$|X| = \int^{n\in\Delta} X_n \times \Delta^n$$

**Example 3.**

$$\begin{tikzcd} \Delta \arrow[r, "\widetilde{sd}(\cdot)"] \arrow[dr, hook, "\mathcal{L}'"'] & sSet \\ & sSet \arrow[u, dashed, "\mathrm{Lan}_{\mathcal{L}'} \widetilde{sd}(\cdot) = sd"'] \end{tikzcd}$$

$sd\, \Delta^n$

$$\begin{tikzcd} & \{1\} \arrow[dl] \arrow[d] \arrow[dr] & \\ \{0,1\} \arrow[r] & \{0,1,2\} & \{1,2\} \arrow[l] \\ \{0\} \arrow[u] \arrow[r] \arrow[ur] & \{0,2\} \arrow[u] & \{2\} \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd}$$

$[2]$

$sd = \mathrm{Lan}_{\mathcal{L}'} \widetilde{sd}(\cdot)$ $\qquad$ $sd : sSet \underset{\perp}{\rightleftarrows} sSet : ex$

$sd\, \Delta^n \to \Delta^n$ is last vertex map

$sd \Rightarrow id$

colim of the $\{\eta_{ex^n}\}$ defines a functor $ex^\infty$

**Functor tensor product**

$\mathcal{M}$ is simpl. enr. tens. and cotens

$F : \mathcal{D} \to \mathcal{M}$, $\quad G : \mathcal{D}^{op} \to \mathcal{V}$

$$G \underset{\mathcal{D}}{\otimes} F := \int^{\mathcal{D}} G \otimes F = \mathrm{coeq}\left[ \coprod_{f : d \to d'} Gd' \otimes Fd \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{d} Gd \otimes Fd \right]$$

**Example.** $A : R^{op} \to \underline{Ab}$ this is a $Ab$-functor

$B : R \to \underline{Ab}$ $\qquad$ $A \underset{\mathbb{Z}}{\otimes} B : R^{op} \times R \to \underline{Ab}$

$$A \underset{R}{\otimes} B = \int^{R} A \underset{\mathbb{Z}}{\otimes} B$$

**Example.** $* \otimes_{\mathcal{D}} F = \operatorname{colim} F$

$* : \mathcal{D}^{op} \longrightarrow sSet$ – const funct. to the term obj.

**Example** $\mathrm{Lan}_K F(d) = \mathcal{D}(K-, d) \otimes_{\mathcal{C}} F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed] & \end{tikzcd}$$

If $X : sSet$ then $|X| = \int^{n \in \Delta} X_n \times \Delta^n \cong X_\bullet \otimes_{\Delta} \Delta^\bullet$

**Example** $X$ is bisimpl. set, i.e. $X : \Delta^{op} \longrightarrow sSet$

then $|X| \in sSet$, $|X| = \mathrm{Diag}(X)$

$$|X|_k = \int^{n \in \Delta} X[n,k] \times \Delta(k,[n]) = X(k,k)$$

## Homotopical categories

**Def.** A left deform. on a homotopical cat $\mathcal{M}$ consists of an endofunctor $Q$ together with a nat. weak equiv:

$$q : Q \overset{\sim}{\Rightarrow} 1$$

**Def.** A left def. for a functor $F : \mathcal{M} \to \mathcal{N}$ consists of left deform. for $\mathcal{M}$, s.t. $F$ is homotopical on an associated subcat of cofib. objects

**Theorem.** If $F : \mathcal{M} \to \mathcal{N}$ has a left deformation $q : Q \overset{\sim}{\Rightarrow} 1$ then $\mathbb{L}F := FQ$ is a left derived functor of $F$

**Def.** Two-sided bar-complex is a simpl. obj $B_\bullet(G, \mathcal{D}, F) \in \mathcal{M}$ for $F: \mathcal{D} \to \mathcal{M}$, $G: \mathcal{D}^{op} \to sSet$, $\mathcal{M}$ is simpl. enr. tens. and cotens.

$$B_n(G, \mathcal{D}, F) = \coprod_{\vec{d}:[n]\to\mathcal{D}} Gd_n \otimes Fd_0$$

$\vec{d}:[n] \to \mathcal{D} \rightsquigarrow d_0 \to d_1 \to \dots \to d_n$ in $\mathcal{D}$

**Example** $B(*, D, *) = N\mathcal{D}$

$\mathcal{M} = Set$, $\mathcal{V} = Set$, $F = *$, $G = *$

**Def.** $B(G, \mathcal{D}, F) = |B_\bullet(G, \mathcal{D}, F)| =$

$$= \Delta^\bullet \otimes_{\Delta^{op}} B_\bullet(G, D, F)$$

$\Delta^n \to * \rightsquigarrow$ nat. transf. $\Delta^\bullet \Rightarrow *$

$$- \otimes_{\Delta^{op}} B_\bullet(G, D, F) \rightsquigarrow \begin{array}{ccc} B(G, D, F) & \to & G \otimes_D F \\ \| & & \| \text{ Exercise} \\ |B_\bullet(G, D, F)| & & \operatorname{colim} B_\bullet(G, D, F) \end{array}$$

**Example.** $BG = |B_\bullet(*, G, *)|$

**Theorem.** Let $\mathcal{M}$ be simpl. mod. cat. Then $B(\mathcal{D}, \mathcal{D}, Q-): \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{D}}$, $B(\mathcal{D}, \mathcal{D}, Q-) \overset{\varepsilon_Q}{\Rightarrow} Q \xrightarrow{q} 1$

is a left deformation for colimit: $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}$

What is $\varepsilon_Q$? $\mathcal{B}(\mathcal{D}, \mathcal{D}, F) \longrightarrow \operatorname{colim} \mathcal{B}_\bullet(\mathcal{D}, \mathcal{D}, F) \cong$

$$\cong \mathcal{D}(-, d) \underset{\mathcal{D}}{\otimes} F = \int^{d \in \mathcal{D}} \mathcal{D}(d', d) \otimes Fd' \cong Fd$$

$$\varepsilon_Q : \mathcal{B}(\mathcal{D}, \mathcal{D}, F) \longrightarrow F$$

Corollary If $\mathcal{M}$ is simpl. mod cat and $\mathcal{D}$ is any small cat then

$$\underset{\mathcal{D}}{\operatorname{hocolim}} := \mathbb{L} \underset{\mathcal{D}}{\operatorname{colim}} \cong \mathcal{B}(*, \mathcal{D}, Q-)$$

$$\underset{\mathcal{D}}{\operatorname{holim}} := \mathbb{R} \underset{\mathcal{D}}{\lim} \cong \mathcal{C}(*, \mathcal{D}, R-)$$

Proof of cor.

$$\mathbb{L} \underset{\mathcal{D}}{\operatorname{colim}}(-) \cong \underset{\mathcal{D}}{\operatorname{colim}} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-) \cong * \underset{\mathcal{D}}{\otimes} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-)$$

$$* \underset{\mathcal{D}}{\otimes} F \cong \int^{d \in \mathcal{D}} *(d) \otimes Fd \cong \operatorname{colim} F$$

mutted coend

$$* \underset{\mathcal{D}}{\otimes} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-) \cong \mathcal{B}(*, \mathcal{D}, Q-)$$

$\triangleleft$

So, $\mathbb{L} \underset{\mathcal{D}}{\operatorname{colim}}(-) \cong \mathcal{B}(*, \mathcal{D}, Q-)$

## Proof the main theorem.

$\varepsilon : B(\mathcal{D}, \mathcal{D}, F) \Rightarrow F$ is pointwise weak equiv., as $\Delta^n \to *$ was a weak equiv

Hence, the comp. $B(\mathcal{D}, \mathcal{D}, F) \xrightarrow{\varepsilon Q} Q \xrightarrow{q} 1$ is a left deform on $\mathcal{M}^{\mathcal{D}}$

What do we want?

$\boxed{\varepsilon \circ q \text{ defines a left deform. of colim}}$ ?

It remains to verify that colim preserves weak equiv. between diagrams in full subcat. of $\mathcal{M}^{\mathcal{D}}$ which contains the image of $B(\mathcal{D}, \mathcal{D}, Q-)$

**Lemma.** $\exists (Q, q)$ is a left deform. on $\mathcal{M}$, $F : \mathcal{M} \to \mathcal{N}$. If
- $FQ$ is homotopical
- $FqQ : FQ^2 \Rightarrow FQ$ is nat. weak equiv.

then $(Q, q)$ is a left deform. for $F$

▷ Let $\varphi$ be a map $\varphi : M_1 \to M_2$

$Q\varphi \in \mathcal{W} \overset{?}{\Longrightarrow} FQ\varphi \in \mathcal{W}$

We know $FQ(Q\varphi) \in \mathcal{W}$ as $FQ$ is homotopical

$\subset \mathcal{W}$

$$\begin{tikzcd} FQ^2M_1 \arrow[r, "FQ^2\varphi"] \arrow[d, "\tau_{M_1}"', "\mathcal{W}\ni"] & FQ^2M_2 \arrow[d, "\tau_{M_2}\in\mathcal{W}"] \\ FQM_1 \arrow[r, "FQ\varphi"] & FQM_2 \end{tikzcd}$$

$\Rightarrow$ from 3-out-of-2 $FQ\varphi \in \mathcal{W}$ $\lhd$

It remains to prove that the functor $B(*, \mathcal{D}, Q-)$ is homotopical and that the composite

$$B(\mathcal{D},\mathcal{D},QB(\mathcal{D},\mathcal{D},QF)) \xrightarrow{\operatorname{colim}_{\mathcal{D}}\varepsilon} \operatorname{colim}_{\mathcal{D}} QB(\mathcal{D},\mathcal{D},F)) \to$$
$$\xrightarrow{\operatorname{colim} q} \operatorname{colim}_{\mathcal{D}} B(\mathcal{D},\mathcal{D},QF)$$

**Def.** Simpl. obj. is called Reedy cofibrant if $\forall$ inclusion from degenerate $n$-simplices into $n$-simplex is pointwise cofibration

(latching object)

**Lemma.** $B_\bullet(*, \mathcal{D}, F)$ is Reedy cofibrant, when $\mathcal{D}$ is a small cat, $\mathcal{M}$ is simpl. mod. cat., $F: \mathcal{D} \to \mathcal{M}$ is a pointwise cofibration.

$\rhd$ $B_n(*, \mathcal{D}, F)$ is $\operatorname{colim}_{(N\mathcal{D})_n} Fd_0$

$n$th latch. obj. sits inside $B_n(*, \mathcal{D}, F)$

But cofibr. are closed under colims and pushouts $\triangleleft$

**Theorem.** $\mathcal{M}$ is simpl. mod. cat.

$$|-| : \mathcal{M}^{\Delta^{op}} \longrightarrow \mathcal{M}$$

is left Quillen with resp. to the Reedy model struc.

**Corollary.** The functor $B(G, \mathcal{D}, -)$ preserves weak equivalences between pointwise cofibr. objects

$\Rightarrow B(*, \mathcal{D}, Q-)$ is homotopical

Now we want to prove that the comp.

$$\bullet \xrightarrow{\operatorname{colim}_{\mathcal{D}} \varepsilon} \bullet \xrightarrow{\operatorname{colim}_{\mathcal{D}} q} \bullet \text{ is weak equiv } (?)$$

$$\begin{tikzcd}
B(\mathcal{D},\mathcal{D}, QB(\mathcal{D},\mathcal{D},QF)) \arrow[r, "\varepsilon_{QB} \in W"] \arrow[d, "B(\mathcal{D},\mathcal{D},q)"', "W"] & QB(\mathcal{D},\mathcal{D},QF) \arrow[d, "q \in W"] \\
B(\mathcal{D},\mathcal{D}, B(\mathcal{D},\mathcal{D},QF)) \arrow[r, "\varepsilon_B \overset{?}{\in} W"'] & B(\mathcal{D},\mathcal{D},QF)
\end{tikzcd}$$

$$\begin{tikzcd}
* \otimes B(\mathcal{D},\mathcal{D},B(\mathcal{D},\mathcal{D},F)) \arrow[r, "* \otimes_{\mathcal{D}} \varepsilon \in W"] \arrow[d, "L"'] & * \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) \arrow[d, "\cong"] \\
* \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},\mathcal{D}) \otimes B(\mathcal{D},\mathcal{D},F) \arrow[r, "* \otimes \varepsilon \otimes B(\mathcal{D},\mathcal{D},F)"] \arrow[d, "L'"'] & * \otimes \mathcal{D} \otimes B(\mathcal{D},\mathcal{D},F) \arrow[d, "\cong"] \\
B(*,\mathcal{D},\mathcal{D}) \otimes B(\mathcal{D},\mathcal{D},F) \arrow[r, "\varepsilon \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F)"] \arrow[d, "L"'] & * \otimes B(\mathcal{D},\mathcal{D},F) \arrow[d, "\cong"] \\
\, & \,
\end{tikzcd}$$

$$\mathcal{B}(\mathcal{B}(*,\mathcal{D},\mathcal{D}),\mathcal{D},F) \xrightarrow{\mathcal{B}(\varepsilon,\mathcal{D},F)} \mathcal{B}(*,\mathcal{D},F) \quad \triangleleft$$

$\uparrow$ $w$ from

$d\in\mathcal{D}$

**Theorem.** Let $F:\mathcal{D}\to\mathcal{M}$

$$\mathcal{B}(*,\mathcal{D},F) \cong N(-/\mathcal{D}) \otimes_{\mathcal{D}} F = \int^{d\in\mathcal{D}} N(d/\mathcal{D}) \otimes Fd$$

$$\mathcal{C}(*,\mathcal{D},F) \cong \int_{d\in\mathcal{D}} \mathcal{D}(N(\mathcal{D}/d), Fd)$$

$\triangleright$ Exercise: Check that $\mathcal{B}(*,\mathcal{D},\underbrace{\mathcal{D}}_{\mathcal{D}(d,-)}) \cong N(-/\mathcal{D})$

Then

$$N(-/\mathcal{D}) \otimes_{\mathcal{D}} F \cong \mathcal{B}(*,\mathcal{D},\mathcal{D}) \otimes_{\mathcal{D}} F \cong$$

$$\cong \mathcal{B}(*,\mathcal{D},\mathcal{D}\otimes_{\mathcal{D}} F) \underset{\text{ninja-Yoneda lemma}}{\cong} \mathcal{B}(*,\mathcal{D},F) \quad \triangleleft$$