

Def. Nerve Realization Context is a functor

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$\swarrow$  small  
 $\nwarrow$  locally small

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \mathcal{L} \searrow & \eta \Downarrow & \uparrow \bar{F} \\ & [\mathcal{C}^{\text{op}}, \text{Set}] & \end{array}$$

Theorem (Univ. property of  $[\mathcal{C}^{\text{op}}, \text{Set}]$ )

1) There is  $\text{Lan}_{\mathcal{L}'_{\mathcal{C}}} F \dashv \text{Lan}_{\mathcal{L}'_{\mathcal{C}}} \dashv \text{L}'_{\mathcal{C}}$ ,

the unit of this adjunction is invertible

2) The ess. image of  $\text{Lan}_{\mathcal{L}'_{\mathcal{C}}}$  consists of those

$F: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$  that preserves all colimits

3) If  $\mathcal{D} = [\mathcal{C}^{\text{op}}, \text{Set}]$ , this ess. image is equivalent to the subset of left adjoints  $F: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow [\mathcal{C}^{\text{op}}, \text{Set}]$

Proof. 1)  $\forall F: \mathcal{C} \rightarrow \mathcal{D}$  extends univ. to  $\bar{F}: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$   
(up to isomorphism)

$\exists$  an adjunction  $\text{Lan}_{\mathcal{L}'_{\mathcal{C}}} \xrightleftharpoons{\eta} \text{L}'_{\mathcal{C}}$   
that has invertible unit  $\eta$

$\bar{F} \Rightarrow (F \mapsto \bar{F})$  is a functor

$\forall G: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$

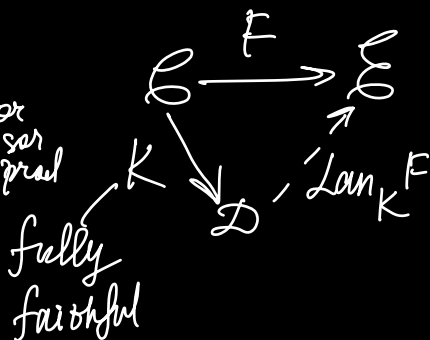
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G \circ \mathcal{L}'} & \mathcal{D} \\ \mathcal{L} \searrow & \eta \Downarrow & \uparrow \bar{F} \\ & [\mathcal{C}^{\text{op}}, \text{Set}] & \end{array} \quad \left( \begin{array}{c} \uparrow \\ \Rightarrow \\ \uparrow \end{array} \right) G$$

$\exists! \varepsilon: \text{Lan}_{\mathcal{L}'}(G\mathcal{L}) \Rightarrow G$  is the counit of our adj

$$\eta: \text{Lan}_{\mathcal{L}'} G \circ \mathcal{L} \cong G$$

Lemma.  $\text{Lan}_{K \text{ cEB}_0} F \circ K \cong F$

$$\triangleright \text{Lan}_K F(d) = \int_{\mathcal{D}(Kc, d)} Fc$$



$$(\text{Lan}_K F)(K(c)) = \int_{x \in \mathcal{B}_0} \mathcal{D}(Kx, Kc) \cdot Fx \cong$$

$$\cong \int_{x \in \mathcal{B}_0} \mathcal{B}(x, c) \cdot Fx \cong Fc$$

2) Suppose that  $G: [\mathcal{B}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$  is continuous  $\Rightarrow$

$\Rightarrow (\varepsilon_G: \text{Lan}_{\mathcal{L}'_{\mathcal{B}}} (G \circ \mathcal{L}'_{\mathcal{B}}) \rightarrow G)$  is invertible

$$\begin{aligned} \text{Lan}_{\mathcal{L}'_{\mathcal{B}}} (G \circ \mathcal{L}'_{\mathcal{B}})(P) &\cong \int_{c \in \mathcal{B}} [\mathcal{B}^{\text{op}}, \text{Set}](\mathcal{L}'c, P) \cdot G\mathcal{L}'c \cong \\ &\cong \int_{c \in \mathcal{B}} P_c \cdot G\mathcal{L}'c \xrightarrow{\sim} G\left(\int_{c \in \mathcal{B}} P_c \cdot \mathcal{L}'c\right) \cong GP \end{aligned}$$

Vice versa, if we have  $G \cong \text{Lan}_{\mathcal{L}'}(G\mathcal{L})$  then

$$G\left(\text{colim}_J P_j\right) \cong \int_{c \in \mathcal{B}} \text{colim}_J (P_j)c \cdot G(\text{hom}(-, c)) \cong$$

$$P: J \rightarrow [\mathcal{B}^{\text{op}}, \text{Set}] \cong \text{colim}_J \underbrace{\int_{c \in \mathcal{B}} (P_j)c \cdot G(\text{hom}(-, c))}_{(\text{Lan}_{\mathcal{L}'}(G\mathcal{L}))(P_j)} \cong$$

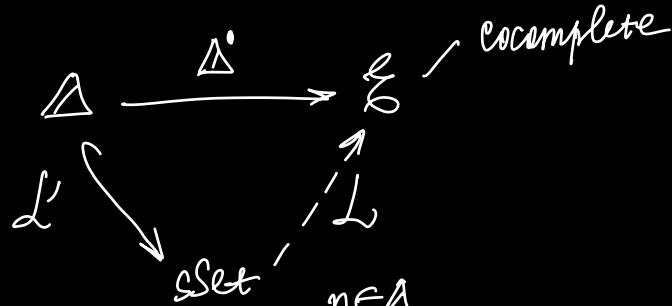
$$\cong \operatorname{colim}_J G(P_j)$$

3)  $L: [\mathcal{E}^{\text{op}}, \text{Set}] \rightarrow [\mathcal{E}^{\text{op}}, \text{Set}]$  — continuous functors

Then it has a right adj given by

$$\int^{C \in \mathcal{E}} [\mathcal{E}^{\text{op}}, \text{Set}](\mathcal{L}C, X) \cdot C \quad \triangleleft$$

Example



$$\mathcal{L}X := \int^{n \in \Delta} \text{sSet}(\Delta^n, X) \cdot \Delta^n \cong \int X_n \cdot \Delta^n \cong \operatorname{coeq} \left[ \coprod_{[n] \rightarrow [m]} X_n \cdot \Delta^n \rightarrow \coprod_{[n] \in \Delta} X_n \right]$$

We know that  $\mathcal{L}\Delta^n \cong \Delta^n$

$\mathcal{L}$  is a colimit  $\Rightarrow \mathcal{L}$  commutes with colimits  $\rightarrow \mathcal{L}$  has a right adj.  $R: \mathcal{E} \rightarrow \text{sSet}$

$$(Re)_n \cong \text{sSet}(\Delta^n, Re) \cong \mathcal{E}(\mathcal{L}\Delta^n, e) \cong \mathcal{E}(\Delta^n, e)$$

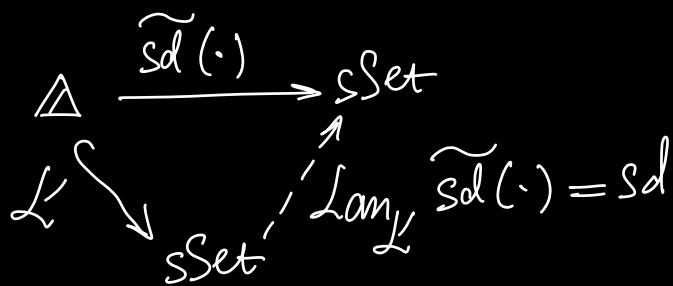
$\Rightarrow n$ -simpl. of  $Re$  are maps in  $\mathcal{E}$  from  $\Delta^n$  to  $e$

Example 2.  $\Delta \rightarrow \text{Top}$ , s.t.  $\Delta^n$  is the stand. top.  $n$ -simplex

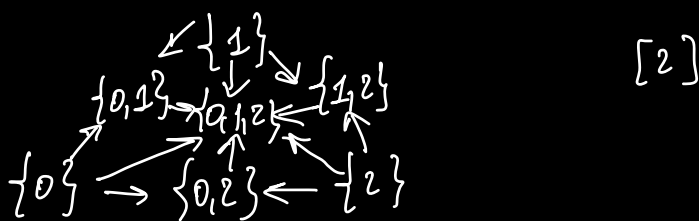
$$|-| : \int^{n \in \Delta^n} \text{sSet} \rightarrow \text{Top} : S$$

$$|X| = \int X_n \times \Delta^n$$

Example 3.



$sd \Delta^n$



$$sd = \text{Lan}_{\downarrow} \widetilde{sd}(\cdot) \quad sd: sSet \rightleftarrows sSet: ex$$

$sd \Delta^n \rightarrow \Delta^n$  is last vertex map

$$sd \Rightarrow id$$

colim of the  $\{\eta_{ex^n}\}$  defines a functor  $ex^\infty$

Functor tensor product

$$F: \mathcal{D} \rightarrow \mathcal{M}, \quad G: \mathcal{D}^{op} \rightarrow \mathcal{V}$$

$\mathcal{M}$  is simpl. enr. ten. and exten.

$$G \otimes F := \int_{\mathcal{D}} G \otimes F = \text{coeq} \left[ \int_{fid \rightarrow d'} Gd' \otimes Fd \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{f \times d} \end{array} \int_{Gd \otimes Fd} Gd \otimes Fd \right]$$

Example.  $A: R^{op} \rightarrow \underline{Ab}$

this is a  $Ab$ -functor

$$B: R \rightarrow \underline{Ab}$$

$$A \otimes_{\mathbb{Z}} B: R^{op} \times R \rightarrow \underline{Ab}$$

$$A \otimes_R B = \int_{\mathbb{Z}} A \otimes_{\mathbb{Z}} B$$

Example.  $* \otimes_{\mathcal{D}} F = \text{colim } F$

$*: \mathcal{D}^{\text{op}} \rightarrow \text{sSet}$  - const funct. to the term obj.

Example  $\text{Lan}_K F(d) = \mathcal{D}(K-, d) \otimes_{\mathcal{L}} F$

If  $X: \text{sSet} \rightarrow \text{sSet}$  then  $|X| = \int_{n \in \Delta} X_n \times \Delta^n \cong X_{\Delta} \otimes_{\Delta} \Delta$

Example  $X$  is bisimpl. set, i.e.  $X: \Delta^{\text{op}} \rightarrow \text{sSet}$

then  $|X| \in \text{sSet}$ ,  $|X| = \text{Diag}(X)$

$$|X|_k = \int_{n \in \Delta} X(n, k) \times \Delta(k, [n]) = X(k, k)$$

## Homotopical categories

Def. A left deform. in a homotopical cat  $\mathcal{M}$  consists of an endofunctor  $Q$  together with a nat. weak equiv:

$$q: Q \xrightarrow{\sim} 1$$

Def. A left def. for a functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  consists of left deform. for  $\mathcal{M}$ , s.t.  $F$  is homotopical on an associated subcat of cofib. objects

Theorem. If  $F: \mathcal{M} \rightarrow \mathcal{N}$  has a left deformation  $q: Q \xrightarrow{\sim} 1$  then  $\mathbb{L}_L F := FQ$  is a left derived functor of  $F$

Def. Two-sided bar-complex is a simpl. obj  $B_n(G, D, F) \in M$  for  $F: D \rightarrow M, G: D^{\text{op}} \rightarrow \text{Set}$ ,  $M$  is simpl. enr. tens and cotens.

$$B_n(G, D, F) = \left[ \right] G d_n \otimes F d_0$$

$$\vec{d}: [n] \rightarrow D$$

$$\vec{d}: [n] \rightarrow D \rightsquigarrow d_0 \rightarrow d_1 \rightarrow \dots \rightarrow d_n \text{ in } D$$

Example  $B(*, D, *) = ND$   
 $M = \text{Set}, V = \text{Set}, F = *, G = *$

Def.  $B(G, D, F) = |B.(G, D, F)| =$

$$= \Delta \otimes_{\Delta^{\text{op}}} B.(G, D, F)$$

$$\begin{array}{ccc} \Delta^n \rightarrow * \rightsquigarrow \text{nat. transf.} & \Delta \Rightarrow * & \\ \downarrow \otimes_{\Delta^{\text{op}}} B.(G, D, F) \rightsquigarrow & B(G, D, F) \rightarrow G \otimes_{\Delta} F & \\ & \parallel & \parallel \text{Exercise} \\ & |B.(G, D, F)| & \text{colim } B.(G, D, F) \end{array}$$

Example.  $BG = |B.(*, G, *)|$

Theorem. Let  $M$  be simpl. mod. cat. Then  
 $B(D, D, Q-): M^D \rightarrow M^D, B(D, D, Q-) \xrightarrow{\epsilon_Q} Q \xrightarrow{q} 1$

is a left deformation for colimit:  $M^{\mathcal{D}} \rightarrow M$

What is  $\varepsilon_Q$ ?  $\mathcal{B}(\mathcal{D}, \mathcal{D}, F) \rightarrow \text{colim}_{\mathcal{D}} \mathcal{B}_*(\mathcal{D}, \mathcal{D}, F) \cong$   
 $\cong \mathcal{D}(-, d) \otimes_{\mathcal{D}} F = \int_{\mathcal{D}} \mathcal{D}(d', d) \otimes F d' \cong F d$

$\varepsilon_Q: \mathcal{B}(\mathcal{D}, \mathcal{D}, F) \longrightarrow F$

Corollary If  $M$  is simpl. mod cat and  $\mathcal{D}$  is any small cat then

$\text{hocolim}_{\mathcal{D}} := \mathbb{L} \text{colim}_{\mathcal{D}} \cong \mathcal{B}(*, \mathcal{D}, Q-)$

$\text{holim}_{\mathcal{D}} := \mathbb{R} \text{lim}_{\mathcal{D}} \cong \mathcal{C}(*, \mathcal{D}, R-)$

Proof of cor.

$\mathbb{L} \text{colim}_{\mathcal{D}}(-) \cong \text{colim}_{\mathcal{D}} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-) \cong * \otimes_{\mathcal{D}} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-)$

$* \otimes_{\mathcal{D}} F \cong \int_{\mathcal{D}} *(d) \otimes F d \cong \text{colim}_{\mathcal{D}} F$   
mutual cend

$* \otimes_{\mathcal{D}} \mathcal{B}(\mathcal{D}, \mathcal{D}, Q-) \cong \mathcal{B}(*, \mathcal{D}, Q-)$

So,  $\mathbb{L} \text{colim}_{\mathcal{D}}(-) \cong \mathcal{B}(*, \mathcal{D}, Q-)$



Proof the main theorem

$\varepsilon: \mathcal{B}(\mathcal{D}, \mathcal{D}, F) \Rightarrow F$  is pointwise weak equiv, as  $\Delta^n \rightarrow *$  was a weak equiv  
 Hence, the comp.  $\mathcal{B}(\mathcal{D}, \mathcal{D}, F) \xrightarrow{\varepsilon_Q} Q \xrightarrow{q} 1$  is a left deform on  $M^{\mathcal{D}}$

What do we want?

[  $\varepsilon \circ q$  defines a left deform. of colim ] ?

It remains to verify that colim preserves weak equiv. between diagrams in full subcat. of  $M^{\mathcal{D}}$  which contains the image of  $\mathcal{B}(\mathcal{D}, \mathcal{D}, Q-)$

Lemma.  $(Q, q)$  is a left deform. on  $M$ ,  
 $F: M \rightarrow N$ . If  $\bullet$   $FQ$  is homotopical  
 $\bullet$   $FqQ: FQ^2 \Rightarrow FQ$  is nat. weak equiv.

then  $(Q, q)$  is a left deform. for  $F$

$\triangleright$  Let  $\varphi$  be a map  $\varphi: M_1 \rightarrow M_2$

$Q\varphi \in \mathcal{W} \stackrel{?}{\Rightarrow} FQ\varphi \in \mathcal{W}$

We know  $FQ(Q\varphi) \in \mathcal{W}$  as  $FQ$  is homotopical

ccw



$$\begin{array}{ccc}
 FQ^2M_1 & \xrightarrow{FQ^2\psi} & FQ^2M_2 \\
 \uparrow \tau_{M_1} & \downarrow & \downarrow \tau_{M_2} \in \mathcal{W} \\
 \mathcal{W} & & \\
 FQM_1 & \xrightarrow{FQ\psi} & FQM_2
 \end{array}$$

$\Rightarrow$  from 3-out-of-2  $FQ\psi \in \mathcal{W}$  ◁

It remains to prove that the functor  $B(*, \mathcal{D}, Q-)$  is homotopical and that the

composite

$$B(\mathcal{D}, \mathcal{D}, QB(\mathcal{D}, \mathcal{D}, QF)) \xrightarrow{\text{colim}_{\mathcal{D}} \epsilon} \text{colim}_{\mathcal{D}} QB(\mathcal{D}, \mathcal{D}, F) \rightarrow \text{colim}_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, QF)$$

Def. Simpl. obj. is called Reedy cofibrant if  $\forall$  inclusion from degenerate  $n$ -simplices into  $n$ -simples  $\leftarrow$  is pointwise cofibration (latching object)

Lemma.  $B(*, \mathcal{D}, F)$  is Reedy cofibrant, when  $\mathcal{D}$  is a small cat,  $\mathcal{M}$  is simpl. mod. cat.,  $F: \mathcal{D} \rightarrow \mathcal{M}$  is a pointwise cofibration.

▷  $B_n(*, \mathcal{D}, F)$  is  $\text{colim}_{(N\mathcal{D})_n} F \circ$   
 $n$ -th latching obj. sits inside  $B_n(*, \mathcal{D}, F)$

But cofibr. are closed under colims and pushouts  $\triangleleft$

Theorem.  $M$  is simpl. mod. cat.

$$|-|: M^{\Delta^{op}} \rightarrow M$$

is left Quillen with resp. to the Reedy model struc.

Corollary. The functor  $B(G, \mathcal{D}, -)$  preserves weak equivalences between pointwise cofibr. objects  $\Rightarrow B(*, \mathcal{D}, Q-)$  is homotopical

Now we want to prove that the comp.

$$\bullet \xrightarrow{\text{colim } \varepsilon} \bullet \xrightarrow{\text{colim } \eta} \bullet \text{ is weak equiv. (?)}$$

$$\begin{array}{ccc}
 B(\mathcal{D}, \mathcal{D}, QB(\mathcal{D}, \mathcal{D}, QF)) & \xrightarrow{\varepsilon_{QB}} & QB(\mathcal{D}, \mathcal{D}, QF) \\
 \downarrow & \searrow & \downarrow q \in W \\
 B(\mathcal{D}, \mathcal{D}, \mathcal{D}) & & B(\mathcal{D}, \mathcal{D}, QF) \\
 \uparrow \omega & & \downarrow \varepsilon_B \in W \\
 B(\mathcal{D}, \mathcal{D}, B(\mathcal{D}, \mathcal{D}, QF)) & \xrightarrow{\varepsilon_B} & B(\mathcal{D}, \mathcal{D}, QF) \\
 \downarrow L & & \downarrow \varepsilon_{\mathcal{D}} \in W \\
 * \otimes B(\mathcal{D}, \mathcal{D}, B(\mathcal{D}, \mathcal{D}, F)) & \xrightarrow{\varepsilon_{\mathcal{D}}} & * \otimes B(\mathcal{D}, \mathcal{D}, F) \\
 \downarrow L' & & \downarrow \cong \\
 * \otimes B(\mathcal{D}, \mathcal{D}, \mathcal{D}) \otimes B(\mathcal{D}, \mathcal{D}, F) & \xrightarrow{\varepsilon_{\mathcal{D}}} & * \otimes \mathcal{D} \otimes B(\mathcal{D}, \mathcal{D}, F) \\
 \downarrow L' & & \downarrow \cong \\
 B(*, \mathcal{D}, \mathcal{D}) \otimes B(\mathcal{D}, \mathcal{D}, F) & \xrightarrow{\varepsilon_{\mathcal{D}}} & * \otimes B(\mathcal{D}, \mathcal{D}, F) \\
 \downarrow L & & \downarrow \cong
 \end{array}$$

$$\mathcal{B}(\mathcal{B}(*, \mathcal{D}, \mathcal{D}), \mathcal{D}, F) \xrightarrow{\mathcal{B}(E, \mathcal{D}, F)} \mathcal{B}(*, \mathcal{D}, F)$$

$\uparrow$   
 $w$  from  $\triangleleft$   
 $d \in \mathcal{D}$

Theorem. Let  $F: \mathcal{D} \rightarrow \mathcal{M}$

$$\mathcal{B}(*, \mathcal{D}, F) \cong \mathcal{N}(-/\mathcal{D}) \otimes_{\mathcal{D}} F = \int_{\mathcal{D}} \mathcal{N}(d/\mathcal{D}) \otimes Fd$$

$$\mathcal{C}(*, \mathcal{D}, F) \cong \int_{\mathcal{D}} \mathcal{D}(\mathcal{N}(\mathcal{D}/d), Fd)$$

$\triangleright$  Exercise: Check that  $\mathcal{B}(*, \mathcal{D}, \mathcal{D}) \cong \mathcal{N}(-/\mathcal{D})$

$\nwarrow \mathcal{D}(d, -)$

Then

$$\mathcal{N}(-/\mathcal{D}) \otimes_{\mathcal{D}} F \cong \mathcal{B}(*, \mathcal{D}, \mathcal{D}) \otimes_{\mathcal{D}} F \cong$$

$$\cong \mathcal{B}(*, \mathcal{D}, \mathcal{D} \otimes_{\mathcal{D}} F) \cong \mathcal{B}(*, \mathcal{D}, F) \quad \triangleleft$$

ninja-Yoneda  
lemma