Adjunctions, limits

&

colimits

Def An adjunction $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ between ∞-categories is comprised of:

- a pair of ∞-categories A & B
- a pair of ∞-functors

$$u : A \to B \quad \& \quad f : B \to A$$

- a pair of ∞-natural transformations

$$\eta : \mathrm{id}_B \Rightarrow uf \quad \& \quad \varepsilon : fu \Rightarrow \mathrm{id}_A$$

So that

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\varepsilon\Downarrow"] & & B \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow\eta"] & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd}$$

$$u \xmapsto{\eta -} ufu \xmapsto{- \varepsilon} u$$

$$\begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & {} & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & {} & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

$$f \xmapsto{- \eta} fuf \xmapsto{\varepsilon -} f$$

Remark. In the the setting of (∞, n)- or (∞,∞)-categories this is "pseudo-style" adjunction

↑

It is not most general adjunctions

But: its relationships to the equivalences

&

to the notions of (co-)limits

Lemma. An adjunction in a 2-category is preserved by any 2-functor

Example. Adjunction between 1-cats

$$\mathrm{Cat} \hookrightarrow \mathrm{hQCat}$$

$$\begin{tikzcd}
A \arrow[r, bend right] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right] \arrow[r, rightsquigarrow] & N(A) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & N(B) \arrow[l, bend right]
\end{tikzcd}$$

regarded as a nerve

Example. Quillen adjunctions

Prop. Given an adjunction $$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$ between ∞-cats

Then

- $\forall$ ∞-cat X
$$\begin{tikzcd} \mathrm{Fun}(X, A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(X, B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

- $\forall$ ∞-cat X
$$\begin{tikzcd} h\mathrm{Fun}(X, A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & h\mathrm{Fun}(X, B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

- $\forall$ simplicial set U
$$\begin{tikzcd} A^U \arrow[r, bend right, "u^U"'] \arrow[r, phantom, "\perp"] & B^U \arrow[l, bend right, "f^U"'] \end{tikzcd}$$

- If the ambient ∞-cosmos is cartesian closed, then $\forall\infty$-cat C
$$\begin{tikzcd} A^C \arrow[r, bend right] \arrow[r, phantom, "\perp"] & B^C \arrow[l, bend right] \end{tikzcd}$$

Prop. Adjunctions compose

$$\begin{tikzcd} C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"] \end{tikzcd}$$

Prop. (uniqueness of adjoints)

- If $f \dashv u$ & $f' \dashv u \Rightarrow f \cong f'$
- Conversely, if $f \dashv u$ and $f \cong f' \Rightarrow f' \dashv u$

Lemma. (minimal adjunction data)

$$\begin{tikzcd} A \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

with $u : A \to B$ the lower arrow:

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \iff \exists \text{ nat. transf. } \begin{array}{l} id_B \Rightarrow uf \\ fu \Rightarrow id_A \end{array} \text{ so that}$$

the triangle equality composites are invertible:

$$f \Rightarrow fuf \Rightarrow f \quad \& \quad u \Rightarrow ufu \Rightarrow u$$

Proof: $\boxed{\Rightarrow}$ Obvious

$\boxed{\Leftarrow}$ • $\eta : id_B \Rightarrow uf$ & $\varepsilon' : fu \Rightarrow id_A$

$$\varphi := f \overset{f\eta}{\Longrightarrow} fuf \overset{\varepsilon' f}{\Longrightarrow} f \quad \& \quad \psi := u \overset{\eta u}{\Longrightarrow} ufu \overset{u\varepsilon'}{\Longrightarrow} u$$

are isomorphisms

- Construct an adjunction $f \dashv u$ with unit η and counit ε'

- $$b \Rightarrow ua \overset{f\circ -}{\rightsquigarrow} fb \Rightarrow fua \overset{\varepsilon'}{\rightsquigarrow} fb \Rightarrow fua \overset{\varepsilon'}{\Rightarrow} a \rightsquigarrow$$
$$fb \Rightarrow a$$

- Let $\begin{matrix} b : X \longrightarrow B \\ a : X \longrightarrow A \end{matrix}$ be fixed generalized elements

$$\begin{tikzcd}[ampersand replacement=\&]
 \& \& \left\{\begin{tikzcd}[ampersand replacement=\&\&] \&\& B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] \&\& \\ X \arrow[ur, "b"] \arrow[rr, "a"'] \&\& \&\& A \end{tikzcd}\right\} \arrow[dd, "u(-)\cdot\eta"] \arrow[dr, "-\cdot\varphi", "\cong"'] \& \\
\left\{\begin{tikzcd}[ampersand replacement=\&\&] X \arrow[rr, "b"] \arrow[dr, "a"'] \&\& \&\& B \\ \&\& A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow"] \&\& \end{tikzcd}\right\} \arrow[urr, "\varepsilon'\cdot f(-)"] \arrow[rrr] \arrow[dr, "\Psi\cdot -"', "\cong"] \& \& \& \left\{\begin{tikzcd}[ampersand replacement=\&\&] \&\& B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] \&\& \\ X \arrow[ur, "b"] \arrow[rr, "a"'] \&\& \&\& A \end{tikzcd}\right\} \\
 \& \left\{\begin{tikzcd}[ampersand replacement=\&\&] X \arrow[rr, "b"] \arrow[dr, "a"'] \&\& \&\& B \\ \&\& A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow"] \&\& \end{tikzcd}\right\} \arrow[urr, "\varepsilon'\cdot f(-)"'] \& \&
\end{tikzcd}$$

- By 2-of-6 property, we have all six morphisms being bijections

- Define

$$\begin{tikzcd}
 & B \arrow[dr, "f"'] \arrow[dr, bend left=60, "f"] \arrow[dr, phantom, shift left=3, "\overset{\cong}{\Downarrow}\varphi^{-1}"] & \\
\varepsilon := \quad A \arrow[ur, "u"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\varepsilon'\Downarrow"] & & A
\end{tikzcd}$$

so that

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] & \arrow[d, phantom, "\Downarrow\varepsilon"] \\
 & A \arrow[ur, "u"] \arrow[rr, equal] & & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[rr, equal] \arrow[d, "\eta\Downarrow"'] & & B \arrow[d, "f"'] \arrow[d, bend left=60, "f"] \arrow[d, phantom, shift left=3, "\overset{\cong}{\Downarrow}\varphi^{-1}"] \\
A \arrow[urr, "u"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\varepsilon'\Downarrow"] & & A
\end{tikzcd}
= f\left(
\begin{tikzcd}
B \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "f"] \arrow[d, phantom, "="] \\
A
\end{tikzcd}\right) f$$

- Form the pasting equality

Ψ $\qquad\eta u$ $\qquad\Psi$

$$\begin{tikzcd}
 & B \arrow[rr, equal] \arrow[dr, "f"] & & B \arrow[rr, equal] \arrow[dr, "f"'] \arrow[dr, bend left=50, "f"] & & B \\
A \arrow[ur, "u"] \arrow[rr, equal] \arrow[urr, phantom, "\Downarrow\varepsilon'"] & & A \arrow[ur, "u"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & & A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow\eta"] \arrow[ul, phantom, "\cong\,\Downarrow\varphi^{-1}"]
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[r, equal] \arrow[dr, "f"] & B \\
A \arrow[ur, "u"] \arrow[rr, equal] \arrow[r, phantom, "\Downarrow\varepsilon'"] & & A \arrow[u, "u"'] \arrow[ul, phantom, "\Downarrow\eta"]
\end{tikzcd}$$

$u\varepsilon$

$$(u\varepsilon\cdot\eta u)\cdot\Psi=\Psi$$

$$\Psi := u \overset{\eta u}{\Longrightarrow} ufu \overset{u\varepsilon'}{\Longrightarrow} u$$

- But Ψ is invertible

So, $u\varepsilon\cdot\eta u=\mathrm{id}_u$

◁

Corollary. (adjoint equivalences) Any equivalence can be promoted to an adjoint equivalence by modifying one of the 2-cells, i.e., the invertible 2-cells in an equivalence can be chosen so as to satisfy the triangle equalities

$$\boxed{f, g \text{ are inverse equiv.} \rightsquigarrow f \dashv g \;\&\; g \dashv f}$$

Proof:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong \Downarrow \alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\cong \Downarrow \beta"] & B \end{tikzcd}$$

$$f \overset{\sim}{\Rightarrow} fgf \overset{\sim}{\Rightarrow} f \quad \& \quad g \overset{\sim}{\Rightarrow} gfg \overset{\sim}{\Rightarrow} g$$

So, by the previous lemma f and g fit into an adjunction ◁

Prop. Adjunctions are equivalence invariant

$u : A \longrightarrow B$ admits a left adjoint $\iff \forall\, A' \simeq A$, $B' \simeq B$ the functor $u' : A' \longrightarrow B'$ admits a left adjoint

∞-cats

Proof: • $\exists\, f \dashv u$ with $A' \simeq A$, $B' \simeq B$ – adjoint equiv.

$$\begin{tikzcd}
A' \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "\sim"'] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & B' \arrow[l, bend right, "\sim"']
\end{tikzcd}$$

• Conversely, if $u' : A' \xrightarrow{\sim} A \xrightarrow{u} B \xrightarrow{\sim} B'$ admits a left adj f' then

$$\begin{tikzcd}
A \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & A' \arrow[l, bend right, "\sim"'] \arrow[r, "\sim"'] & A \arrow[r, "u"'] & B \arrow[r, "\sim"'] & B' \arrow[lll, bend right, "f'"'] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right]
\end{tikzcd}$$

$f \Rightarrow fuf \Rightarrow f$ & $u \Rightarrow ufu \Rightarrow u$

◁

Initial & terminals elements

Def. An initial element in an ∞-cat A is a left adjoint to the unique functor $!: A \to 1$

A terminal element is a right adj.

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

Lemma. (minimal data) To define an initial element in an ∞-cat A, it suffices to specify:

- an element $i: 1 \to A$
- a nat. transf. $$\begin{tikzcd} & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\ A \arrow[ur, "!"] \arrow[rr, equal] & {} & A \end{tikzcd}$$ so that $\varepsilon i : i \Rightarrow i$ in hA is inv.

Proof:

- ∞-cat $1 \in \mathcal{K}$ is 2-terminal in the homotopy 2-cat $h\mathcal{K}$
- $i : 1 \longrightarrow A$ is a section of $! : A \longrightarrow 1$

So, $!i = \mathrm{id}_1$ & $\eta : 1 \xrightarrow{\sim} !i$ is an iso

We have a triangle

$$\begin{tikzcd}
 & A \arrow[rr, equal] \arrow[dr, "!"] & \Downarrow\varepsilon & A & & A \\
1 \arrow[ur, "i"] \arrow[rr, equal] & \eta\Downarrow & 1 \arrow[ur, "i"'] & & = & 1 \arrow[u, bend left, "i"] \arrow[u, bend right, "i"'] \arrow[u, phantom, "="]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, "i"'] & \Downarrow\varepsilon & A \arrow[dr, "i"] & & & A \arrow[d, bend right, "!"'] \arrow[d, bend left, "!"] \arrow[d, phantom, "="] \\
 & 1 \arrow[ur, "!"] \arrow[rr, equal] & \Downarrow\eta & 1 & = & 1
\end{tikzcd}$$

$$i \xmapsto{i\eta} i!i \xmapsto{\varepsilon i} i \qquad ! \xmapsto{\eta !} !i! \xmapsto{!\varepsilon} !$$

$\varepsilon : i! \longrightarrow \mathrm{id}$

$\eta : \mathrm{id} \Rightarrow !i$

this does not

We want:

$\varepsilon i = \mathrm{id}_i$

this holds automatically

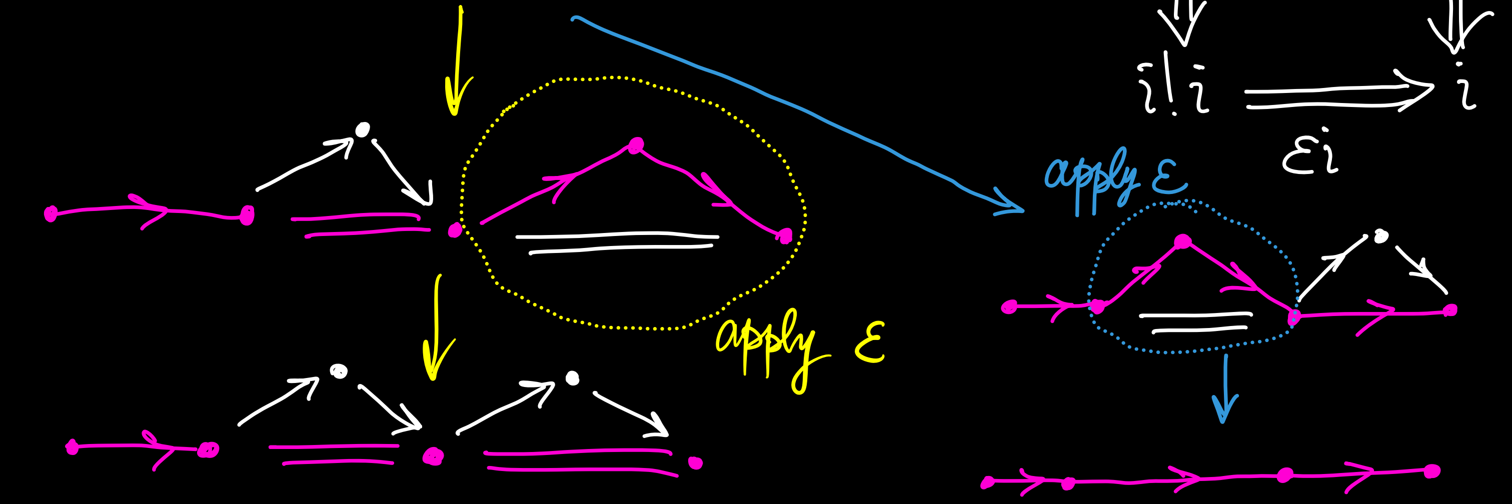
$!\varepsilon(a) = !(a)$

- To do this, it suffices to require an isomorphism

$$\varepsilon i : \bar{i} \cong i \qquad \text{(see the prop. above)}$$

- Consider the horizontal composite and represent it as a vertical composite in two ways:

$$\begin{tikzcd}
 & & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow \varepsilon"] & & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\
1 \arrow[r, "i"] & A \arrow[ur, "!"] \arrow[rr, equal] & & A \arrow[ur, "!"] \arrow[rr, equal] & & A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
i!i!i \arrow[r, Rightarrow, "\varepsilon i!i"] \arrow[d, Rightarrow, "i!\varepsilon i"'] & i!i \arrow[d, Rightarrow, "\varepsilon i"] \\
i!i \arrow[r, Rightarrow, "\varepsilon i"'] & i
\end{tikzcd}$$



- So, $\varepsilon_i \cdot \varepsilon_i = \varepsilon_i \implies \varepsilon_i = id_i$ since ε_i is an isomorphism by the assumption ◁

Lemma. (uniqueness) Any two initial elements in an ∞-cat A are isomorphic in hA and if $a \cong \underset{\substack{\uparrow \\ \text{initial}}}{i}$ in $hA \Rightarrow a$ is initial too

Proof:
- $\forall$ left adjoints i and i' to $! : A \to 1$ are nat. isomorphic
- $\forall\, a : 1 \longrightarrow A$ that is isomorphic to left adj to $! : A \to 1$ is itself a left adjoint
- A functor isomorphism $i \cong i'$ gives $i \cong i'$ in hA ◁

Lemma. An element $i : 1 \to A$ is initial $\iff \forall f : X \to A$

$\exists$ a unique 2-cell with boundary

$$\begin{tikzcd}
 & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow \exists!"] & \\
X \arrow[ur, "!"] \arrow[rr, "f"'] & {} & A
\end{tikzcd}$$

Proof. • $i \dashv ! \rightsquigarrow$

$$\begin{tikzcd}
\mathbb{1} \cong h\mathrm{Fun}(X, 1) \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & h\mathrm{Fun}(X, A) \arrow[l, bend left, "!"]
\end{tikzcd}$$

So, the constant functor $i! : X \to A$ is initial in $h\mathrm{Fun}(X, A)$

• Conversely, if $i : 1 \to A$ satisfies univ. property of the statement

Then apply it to the $\mathrm{id}_A : A \to A$ any by Lemma above

we are done $\lhd$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left=50, "i"] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, phantom, shift right=3ex, "\perp"] & hA \arrow[l, "!" description]
\end{tikzcd}$$

Remark.

- Being homotopy initial is weaker than being initial in the ∞-cat
- But a homotopy initial element in a complete $(\infty, 1)$-cat defines an initial elements

Lemma. If A has an initial element and $A \cong A'$ then A' has an initial element and they are respected by the equivalence up to isomorphism

Proof: $A \cong A' \rightsquigarrow$ an adjoint equivalence

$$\begin{tikzcd}
1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=60] & A \arrow[l, bend left, "!"] \arrow[r, bend left, "\sim"] \arrow[r, phantom, "\perp"] & A' \arrow[l, bend left, "\sim"]
\end{tikzcd}$$

it defines an initial element

By the uniqueness of initial elements, the equivalence $A' \xrightarrow{\sim} A$ preserves initial elements

$\triangleleft$

Limits & colimits

(co-)limits should be interpreted as homotopy ones in ∞-cat

(Co-)limits of a diagram valued inside an ∞-cat A in some ∞-cosmos

→ indexed by a simpl. set J in an ∞-cat A in a generic ∞-cosmos

→ indexed by an ∞-cat J and valued in an ∞-cat A in a cartesian closed ∞-cosmos

Def. (diagram ∞-cat)

For $A \in \mathcal{K}$, $J \in sSet$ or $J \in \mathcal{K}_{cart.closed}$ define a diagram of shape J in A as an element

$$d : 1 \longrightarrow A^J$$

$$\mathrm{sSet}^{op} \times \mathcal{K} \longrightarrow \mathcal{K} \qquad\qquad \mathcal{K}^{op} \times \mathcal{K} \longrightarrow \mathcal{K}$$

$$(J, A) \longmapsto A^J \qquad\qquad (J, A) \longmapsto A^J$$

— bifunctors

Apply bifunctor to $! : J \longrightarrow 1$

$\Delta : A \longrightarrow A^J$ — constant diagram functor

Def. (limit & colimit functor)

$$\begin{tikzcd}
A \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, bend right, "\lim"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & A \arrow[l, "\Delta"']
\end{tikzcd}$$

Lemma. Products or coproducts in an ∞-cat A also define define ones in its homotopy cat hA

Proof: • If J is a set $\Rightarrow A^J \simeq \prod_J A$

$$\begin{tikzcd}
h\mathcal{K} \arrow[r, "h\mathrm{Fun}(1,-)"] & \mathrm{Cat} \\
A \arrow[r, mapsto] & hA
\end{tikzcd}$$

← preserves products

$$h(A^J) \simeq h\Big(\prod_J A\Big) \simeq \prod_J hA \underset{J \text{ is a set}}{\simeq} (hA)^J$$

So,

$$\begin{tikzcd}[column sep=huge]
(hA)^J \simeq h(A^J) \arrow[r, bend left=50, "\mathrm{colim}"] \arrow[r, bend right=50, "\lim"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & hA \arrow[l, "\Delta" description]
\end{tikzcd}$$

Put $J = \emptyset$ $\triangleleft$

The definition is insufficiently general!

Def. (absolute lifting diagram)

Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in a 2-cat

An absolute left lifting of g through f is given by a 1-cell ℓ & 2-cell λ:

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] & & X \arrow[r] \arrow[d] \arrow[dr, phantom, shift left=3, "\exists!\Uparrow\xi"] & B \arrow[d] \\
C \arrow[r, "g"'] & A \arrow[r, phantom, "="] & {} & C \arrow[r] \arrow[ur, "\ell"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A
\end{tikzcd}$$

Any 2-cell χ factors through (ℓ, λ)

An absolute right liftin of g through f :

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"'] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[d, phantom, shift left=4, "\exists!\Downarrow\xi"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=5, "\Downarrow\rho"] & A \end{tikzcd}$$

"Absolute" means

Lemma. Left (right) lifting diagrams are stable under restriction of their domain object:

If (ℓ, λ) - abs. left lifting of g through, then $\forall\ c : X \to C$ $(\ell c, \lambda c)$ defines an absolute left lifting of gc through f

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"'] & A \end{tikzcd}$$

Example: $\mathrm{id}_B \Rightarrow uf$ is the unit of $f \dashv u$

$$\Updownarrow$$

(f, η) defines an absolute left lifting diagram

$$\begin{tikzcd}
 & A \arrow[d] \\
B \arrow[ur, "f"] \arrow[r, equal, "\mathrm{id}"'] \arrow[ur, phantom, shift right=4, "\Uparrow\eta"] & B
\end{tikzcd}$$

Dually for $\varepsilon : fu \Rightarrow \mathrm{id}_A$

$$\begin{tikzcd}
 & B \arrow[d] \\
A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=4, "\Downarrow\varepsilon"] & A
\end{tikzcd}$$

Proof:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d] & = & X \arrow[r] \arrow[d] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow\beta"] & B \arrow[d] \\
A \arrow[r, equal] & A & & A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=4, "\Downarrow\varepsilon"] & A
\end{tikzcd}$$

$\alpha : fb \Rightarrow a \rightsquigarrow \beta : b \Rightarrow ua$

$$\begin{tikzcd}
\beta \arrow[rr, leftrightarrow] \arrow[d, phantom, sloped, "\ni"] & & \alpha \arrow[d, phantom, sloped, "\ni"] \\
hFun(X,B) \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\perp"] & & hFun(X,A) \arrow[ll, bend left, "u_*"]
\end{tikzcd}$$

If $f \dashv u$ then it is an adjunction and (u, ε) defines an absolute right lifting of id_A through f

Conversely, the unit & triangle identities of an adj. can be extracted from the univ. prop. of the right lifting diagram ◿

In particular, $colim \dashv \Delta \dashv lim$ define

$$\begin{tikzcd}
& A \arrow[d, "\Delta"] \\
A^J \arrow[ur, "colim"] \arrow[r, equal] \arrow[r, phantom, shift left=4, "\Uparrow\eta"] & A^J
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
& A \arrow[d] \\
A^J \arrow[ur] \arrow[r, equal] \arrow[r, phantom, shift left=4, "\Downarrow\varepsilon"] & A^J
\end{tikzcd}$$

These univ. properties are preserved under restriction. It motivates

Def. (limit & colimit) A colimit of a family of diagrams $d : D \longrightarrow A^J$ of shape J in an ∞-cat A is given by an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\operatorname{colim} d"] \arrow[r] \arrow[r, phantom, shift left=3ex, "\Uparrow \eta"] & A^J \end{tikzcd}$$

comprised of a generalized element $\operatorname{colim} d : D \to A$ &

a colimit cone $\eta : d \Rightarrow \Delta \operatorname{colim} d$

Dually,

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \varepsilon"] & A^J \end{tikzcd}$$

a limit of a family of diagrams $d : D \to A^J$ of shape J in an ∞-cat A

$\varepsilon : \Delta \lim d \Rightarrow d$ — a limit cone

Example An initial element $i : 1 \to A$ can be regarded as a colimit of the empty diagram

The constant diagram functor:

$$! : A \longrightarrow A^{J} = A^{\emptyset} = 1$$

$$\begin{tikzcd}
& A \arrow[d, "!"] \\
1 \arrow[ur, "i"] \arrow[r, equal] \arrow[ur, phantom, "\parallel"', near end, shift left=2ex] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow \chi"] & A \arrow[d, "!"] \\
1 \arrow[r, equal] & 1
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "!"'] \arrow[r, phantom, "\exists! \Uparrow \xi"', shift right=3ex] & A \arrow[d, "!"] \\
1 \arrow[ur, "i"] \arrow[r, equal] \arrow[r, phantom, "\parallel", shift left=3ex, near end] & 1
\end{tikzcd}$$

We want: an initial element defines an absolute left lifting diagram whose 2-cell is identity

But the existense & uniqueness of ξ follow from initiality of $i : 1 \to A$ among all generalized elements $f : X \to A$

Example (exercise) In a cartesian closed ∞-cosmos, $i : 1 \to A$ can be regarded as a limit of $\mathrm{id}_A : A \to A$

Theorem Right adjoints preserve limits

Proof: •] A admits limits of $d : 1 \longrightarrow A^J$

$$\begin{tikzcd}
 & A \arrow[d, "\Delta"] \\
1 \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\lambda", near end, shift right=3] & A^J
\end{tikzcd}$$

- $f \dashv u \rightsquigarrow f^J \dashv u^J$
- Show that

$$\begin{tikzcd}
 & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
1 \arrow[ur, "\lim d"] \arrow[r] \arrow[ur, phantom, "\Downarrow\lambda", near end, shift right=3] & A^J \arrow[r] & B^J
\end{tikzcd}$$

is again an absolute right lifting diagram

- Consider

$$\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d, "!"'] & {} \arrow[d, phantom, "\Downarrow \chi"] & B \arrow[d, "\Delta"] \\
1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"'] & B^J
\end{tikzcd}$$

- Add a square to the right hand side, composing with f

$$\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d, "!"'] & {} \arrow[d, phantom, "\Downarrow \chi"] & B \arrow[r, "f"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\
1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"'] \arrow[rr, bend right=50, equal, "\Downarrow \varepsilon^J"'] & B^J \arrow[r, "f^J"] & A^J
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "\exists! \Downarrow \xi"] & B \arrow[r, "f"] & A \arrow[d, "\Delta"] \\
1 \arrow[urr, "\lim d" description] \arrow[rr, "d"'] & {} \arrow[r, phantom, "\Downarrow \lambda"] & A^J
\end{tikzcd}$$

just compose
&
factorize

- Add a square to the right hand side, composing with u

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "!"'] \arrow[dr, phantom, "\exists!\Downarrow\zeta"] & B \arrow[r, "f"] \arrow[rr, bend left=50, equal, "\Downarrow\eta"'] & A \arrow[r, "u"'] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
1 \arrow[rr, "d"'] \arrow[urr, "\lim d"] & {} \arrow[ur, phantom, "\Downarrow\lambda"] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"'] & B^{\mathcal{J}}
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d, "!"'] & {} \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[r, "f"'] \arrow[d, "\Delta"] \arrow[rr, bend left=50, equal, "\Downarrow\eta"'] & A \arrow[r, "u"'] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
1 \arrow[r, "d"] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"] \arrow[rr, bend right=50, equal, "\Downarrow\varepsilon^{\mathcal{J}}"] & B^{\mathcal{J}} \arrow[r, "f^{\mathcal{J}}"] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"] & B^{\mathcal{J}}
\end{tikzcd}$$

$$=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d, "!"'] & & B \arrow[rr, equal] \arrow[d, "\Delta"] & & B \arrow[d, "\Delta"] \\
1 \arrow[r, "d"'] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"] \arrow[rr, bend right=50, equal, "\varepsilon^{\mathcal{J}}\Downarrow"] & B^{\mathcal{J}} \arrow[r, "f^{\mathcal{J}}"'] \arrow[rr, bend left=50, equal, "\eta^{\mathcal{J}}\Downarrow"] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"'] & B^{\mathcal{J}}
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d, "!"'] & {} \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[d, "\Delta"] \\
1 \arrow[r, "d"'] & A^{\mathcal{J}} \arrow[r, "u^{\mathcal{J}}"'] & B^{\mathcal{J}}
\end{tikzcd}$$

apply a triangle identity of $f^{\mathcal{J}} \dashv u^{\mathcal{J}}$

So, χ factors through the composite of ζ and η

Uniqueness is left as an exercise ◁

Corollary Equivalences preserve limits

Moreover,

Prop. An equivalence $f: A \xrightarrow{\sim} B$ preserves, reflects and creates limits and colimits

Proof: Prove that f reflects limits and colimits

- Consider a family of diagrams $d: K \to A^J$ that admits limits in B after comp. with $A \cong B$

$$\begin{tikzcd}
 & & B \arrow[d, "\Delta"] \\
K \arrow[urr, "\lim"] \arrow[r, "d"'] & A^J \arrow[r, "\sim"'] \arrow[ur, phantom, "\Downarrow \lambda"] & B^J
\end{tikzcd}$$

- The composite 2-cell

$$\begin{tikzcd}
 & & B \arrow[r, "\sim"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\
K \arrow[r, "d"] \arrow[urr, "\lim"] & A^{\mathcal{J}} \arrow[r, "\sim"'] \arrow[ur, phantom, "\Downarrow\lambda"] \arrow[rrr, bend right=30, equal, "\Downarrow\cong"] & B^{\mathcal{J}} \arrow[r, "\sim"] & A^{\mathcal{J}}
\end{tikzcd}$$

again defines
an absolute
right lifting
diagram

The univ. prop. can be verified ◁

Thank you!