$\mathcal{C}$ – some cat. with a Grothend. top. τ

$$U_\bullet \in (sPre(\mathcal{C}))_0 = PSh(\mathcal{C})^{\Delta^{op}}$$

U_n is a presheaf of sets on $\mathcal{C}$

Def. $U \to V$ is called a hypercover if

every

$$\begin{tikzcd} \partial\Delta[n]\cdot X \arrow[r] \arrow[d] & U \arrow[d] \\ \Delta[n]\cdot X \arrow[r] \arrow[ur, dashed] & V \end{tikzcd} \simeq \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & U(X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & V(X) \end{tikzcd}$$

there exists a solution (σ_i) after refining to some $\{X_i \to X\}$

$$\forall i \quad \begin{tikzcd} \partial\Delta^n[n] \arrow[r] \arrow[d] & U(X_i) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & V(X_i) \end{tikzcd}$$

Def. $U \xrightarrow{f} V \in sPSh(\mathcal{C})$ – a hypercover

if $\forall n \in \mathbb{N}$

$$U_n \longrightarrow (\mathrm{cosk}_{n-1} U)_n \times_{(\mathrm{cosk}_{n-1} V)_n} V_n$$

in $PSh(\mathcal{C})$ are local epimorphisms

$$\begin{tikzcd} U \arrow[rrr] \arrow[dr, dashed] \arrow[dd] & & & V \arrow[dd] \\ & \bullet \arrow[urr] \arrow[dl] \arrow[dr, phantom, "\lrcorner"] & & \\ \operatorname{cosk}_{n-1} U \arrow[rrr] & & & \operatorname{cosk}_{n-1} V \end{tikzcd}$$

Remark. If $Sh(\mathcal{C})$ has enough points then $f: U \to V$ in $sSh(\mathcal{C})$ is a hypercover if all stalks are acyclic Kan fibrations

Recall, a point x of a topos E is a geom. morphism

$$x: Set \underset{x_*}{\overset{x^*}{\rightleftarrows}} E$$

A topos E has enough points if isomorphy can be tested by stalks, i.e. if $\{x^*\}$ are jointly conservatives

$$\underbrace{F(g) - \text{iso} \Rightarrow g - \text{iso}}_{}$$

F is conservative

Example: $Sh(X)$ where X is a topolog space

The main example of hypercover: $(\mathcal{C}, \tau)$

Suppose that $U \to V$ is a τ-cover

$$\check{U}_n = \underbrace{U \times_V \dots \times_V U}_{(n+1) \text{ times}}$$

$\check{U} \longrightarrow V$ is called a Čech hypercover

Theorem. The Bousfield localization of $sPSh(\mathcal{C})$ with respect to the class of Čech hypercovers

$$\check{U}_\cdot \longrightarrow V$$

exists: $\mathcal{L}_\tau sPSh(\mathcal{C})$

Proof: $\forall M$ – a left proper comb. simpl. mod. cat. and I – a set of morphisms in M $\mathcal{L}_I M$ exists and it is a simpl. mod. cat.

Čech τ-hypercovers form a set as $\mathcal{C}$ is small $\triangleleft$

Def. $X \in sPSh(\mathcal{C})$, $V \in \mathcal{C}_0$

$x \in X(V)$ – a basepoint

$$\pi_n(X, x)(U) := \pi_n(X(U), f^*(x))$$

for $f: U \longrightarrow V$ an object $\mathcal{C}/_V$

Let $\pi_n^\tau(X, x)$ be the sheaf.-n of $\pi_n(X, x)$

in the τ-topology restricted to $\mathcal{C}/_V$

Let W_τ be $\Big\{ s : X \to Y$ in $sPSh(\mathcal{C})$ s.t. $s_* : \pi_n^\tau(X(V), x) \xrightarrow{\cong} \pi_n^\tau(Y(V), s(x))$, $n \geq 1$, $\forall V$ $\forall$ basepoint $x \in X(V)$; if $n = 0$ then s_* – a bij. $\Big\}$

(W_τ, ObjCof) — a model structure on $sPSh(\mathcal{C})$

the class of WE   the class of objectwise cofibrations

Jardine – Joyal model structure

Theorem. (Dugger – Hollander – Isaksen)

$$\mathcal{L}_\tau\, sPSh(\mathcal{C}) \xrightarrow{\ id\ } sPSh_J(\mathcal{C})$$

Example A τ-sheaf of sets on $\mathcal{C}$, viewed as a presheaf of simpl. set is fibrant

If τ is subcanonical (i.e., every representable presheaf is in fact a sheaf) the Yoneda embedding

$$\mathcal{C} \longrightarrow PSh(\mathcal{C})$$

factors through the category of fibrant objects for the τ-local model cat on $sPSh(\mathcal{C})$.

Nisnevich's topology

S — quasi compact & quasi-separated scheme

Sm_S — the cat of finitely presented smooth schemes over S

Def $\{u_\alpha : X_\alpha \longrightarrow X\}$ — a Nisnevich cover if

- each morphism u_α is étale
- $\forall x \in X \quad \exists \alpha, \exists y \in X_\alpha$ s.t. $u_\alpha(y) = x$

  a point
- $k(x) \cong k(y)$ — a map of residue fields

Example. k — a field of char $\neq 2$ and $a \overset{\neq 0}{\in} k$

$$\mathbb{A}^1 \setminus \{a\} \xrightarrow{x \mapsto x} \mathbb{A}^1$$
$$\mathbb{A}^1 \setminus \{0\} \xrightarrow{x \mapsto x^2} \mathbb{A}^1$$
étale

This étale covering is Nisnevich $\Leftrightarrow$ a is a square in k

Example Zariski covers are in particular Nisnevich covers: e.g., the usual covering of $\mathbb{P}^1$ is Nisnevich

Def.
$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$
is called an elementary distinguished (Nisnevich) square if i – a Zariski open immersion, p – étale

$$p^{-1}(X-U) \longrightarrow X-U \text{ – an iso}$$

Lemma. $\{i : U \to X,\ p : V \to X\}$ is a Nisnevich cover of X in the above setting

Non-Example:
$$\left.\begin{array}{l} \mathbb{A}^1-\{a\} \xrightarrow{x \to x} \mathbb{A}^1 \\ \mathbb{A}^1-\{0\} \xrightarrow{x \to x^2} \mathbb{A}^1 \end{array}\right\}$$
does not come from an elementary dist. square if $a=0$

When $a \neq 0$ it is so.

Def. $\tau = Nis \rightsquigarrow$ we get the Nisnevich-local model category $\underbrace{\mathcal{L}_{Nis}\, sPSh(Sm_S)}_{Spc_S}$

$\mathrm{Fib}(\mathrm{Spc}_S) \overset{\text{def}}{=}$ spaces

A space is a presheaf of Kan complexes on Sm_S which is a sheaf in Nisnevich topology

A tool for verifying Nisnevich fibrancy in practice

Prop. S — noetherian scheme of finite Krull dimension. A simplicial presheaf F on Sm_S is Nisnevich-fibrant $\Leftrightarrow$ $\forall$ elem. dist. square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

the natural map

$$F(X) \longrightarrow F(U) \underset{F(U \times_X V)}{\times} F(V)$$

is a WE of simplicial sets and $F(\emptyset)$ is a final object

The $\mathbb{A}^1$-homotopy category

Def. Let I be the class of maps

$$\mathbb{A}^1 \underset{S}{\times} X \longrightarrow X \quad \text{in } \mathcal{L}_{Nis}\, sPSh(Sm_S)$$

X ranges over all objects of Sm_S

Choose a subset $J \subseteq I$ containing maps

$$\mathbb{A}^1 \underset{S}{\times} X \to X$$

X ranges over a representative of each isomorphism class of Sm_S

Def. The $\mathbb{A}^1$-homotopy theory of S is the left Bousfield localization of $\mathcal{L}_{Nis}\, sPSh(Sm_S)$ with resp. to J:

$$\mathcal{L}_{\mathbb{A}^1} \mathcal{L}_{Nis}\, sPSh(Sm_S)$$

$Ho\big(\underbrace{\mathcal{L}_{\mathbb{A}^1} \mathcal{L}_{Nis}\, sPSh(Sm_S)}_{\overset{\text{def}}{=}\, Spc_S^{\mathbb{A}^1}}\big)$ is called the $\mathbb{A}^1$-homotopy category of S

Prop. The Bousfield localization $Spc_S^{\mathbb{A}^1}$ exists

Remark. A simplicial presheaf $X \in sPSh(Sm_S)$ is $\mathbb{A}^1$-space if it

1. takes values in Kan complexes (i.e., it is fibrant in $sPSh(Sm_S)$)
2. satisfies Nisnevich hyperdescent (i.e., it is fibrant in Spc_S)
3. if $X(U) \longrightarrow X(\mathbb{A}^1 \times_S U)$ is WE of simplicial sets $\forall U \in Sm_S$

$$\begin{array}{ccc} Spc_S^{\mathbb{A}^1} & \rightleftarrows & Spc_{S,*}^{\mathbb{A}^1} \\ X & \longmapsto & X_+ \\ \text{a presheaf} & & \text{the pointed presheaf of spaces obtained by adding a disjoint basepoint} \end{array}$$

Def. The WE in $Spc_S^{\mathbb{A}^1}$ are called $\mathbb{A}^1$-weak equiv. or $\mathbb{A}^1$-local weak equivalences

Def. Let $f, g : X \to Y$ be maps of simplicial presheaves. We say that f, g are $\mathbb{A}^1$-homotopic if $\exists$ a map $H : F \times \mathbb{A}^1 \longrightarrow G$ s.t.

$$H \circ (id_F \times i_0) = f$$
$$H \circ (id_F \times i_1) = g$$

i_0, i_1 are resp. inclusions of points 0 and 1 into $\mathbb{A}^1$