The 2-category of quasi-categories

- The adjunction

$$h : sSet \underset{}{\overset{\perp}{\rightleftarrows}} Cat : N$$

- The counit is an isomorphism

Lemma. The functor $h : sSet \longrightarrow Cat$ preserves finite product

$\triangleright$ • $(h-)\times(h-)$ & $h(- \times -)$ preserve colimits

- sSet & Cat are cartesian closed

- $h\Delta^n \times h\Delta^m \overset{?}{\cong} h(\Delta^n \times \Delta^m)$

- $\Delta^n = N(\tilde{n})$ where $\tilde{n}$ is some category $\qquad hN = \varepsilon$ is iso

- $(h\Delta^n)\times(h\Delta^m) \cong (hN\tilde{n})\times(hN\tilde{m}) \cong \tilde{n}\times\tilde{m} \cong hN(\tilde{n}\times\tilde{m}) \cong h(N\tilde{n}\times N\tilde{m})$ $\triangleleft$

- Hence, h and N are strong monoidal
- $h_* : \mathrm{sCat} \underset{\perp}{\rightleftarrows} 2\mathrm{Cat} : N_*$

Def. $\mathrm{qCat}_\infty \hookrightarrow \underline{\mathrm{sSet}} \rightsquigarrow \underbrace{\mathrm{qCat}_2}_{\text{2-category of quasi-categories}} := h_* \mathrm{qCat}_\infty$

$\mathrm{Ob}(\mathrm{qCat}_2) = \{\text{quasi-categories}\}$

1-cells of $\mathrm{qCat}_2 = \{\text{maps of quasi-categories}\}$

2-cells of $\mathrm{qCat}_2 = \{\text{homotopy classes of homotopies}\}$

2-cell $\alpha : f \Rightarrow g \;\leftrightsquigarrow\;$ 1-simplex $\tilde{\alpha} : f \to g$ in Y^X

$\alpha_1 \sim \alpha_2 \iff \tilde{\alpha}_1 \sim \tilde{\alpha}_2$ as 1-simplices in Y^X

Prop. $qCat_2$ is cartesian closed

- $h : qCat \longrightarrow Cat \rightsquigarrow$ 2-functor $h_* : qCat_2 \longrightarrow \underline{Cat}$

$$
\begin{tikzcd}
h(Y^X) \times hX \cong h(Y^X \times X) \arrow[r, "h(ev)"] \arrow[d, leftrightsquigarrow, "adj"] & hY \\
h(Y^X) \arrow[r] & hY^{hX}
\end{tikzcd}
$$

On hom-cats

- The aim: the category theory of quasi-categories
- $qCat_2$ has finite products (see above)
- 2-limits theory
- Cotensors with the walking arrow category $\mathbb{2}$

Weak limits in $qCat_2$

$$F: qCat_2(A, X^{\mathbb{2}}) \cong h\left((X^{\mathbb{2}})^A\right) \cong h\left((X^A)^{\mathbb{2}}\right) \overset{?}{\cong} \left(h(X^A)\right)^{\mathbb{2}} \cong$$

cotensors commute with internal hom

$$\cong \left(qCat_2(A, X)\right)^{\mathbb{2}}$$

It defines a notion of cotensoring by $\mathbb{2}$

- We require $\bullet \longrightarrow \bullet$

$$h(X^{\mathbb{2}}) \cong (hX)^{\mathbb{2}}$$

- In $qCat_\infty$ $X^{\Delta^1} \in qCat$. Also, $N\mathbb{2} = \Delta^1$ and $h\Delta^1 = \mathbb{2}$

$$h(X^{\Delta^1}) \longrightarrow (hX)^{h\Delta^1} \cong (hX)^{\mathbb{2}}$$ is not iso

- A weak cotensor with $\mathbb{2}$

Lemma (the universal property) The canonical comparison functor

$$h(X^{\Delta^1}) \longrightarrow (hX)^{\mathbb{2}}$$

is surjective on objects, full & conservative

it's smothering

reflects iso

▷ • Surjectivity: every arrow in hX is represented by a 1-simplex in X

• Fullness: find a morphism from $h(X^{\Delta^1})$ to

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, "k"] & \bullet \arrow[d, "g"] \arrow[dl, phantom, "\sim" near start] \\ \bullet \arrow[r, "b"'] \arrow[ur, phantom, "\sim" near start] & \bullet \end{tikzcd}$$

$$\begin{aligned} k &\sim ga \\ k &\sim bf \end{aligned} \quad \text{in } hX$$

We have

$$\Delta^1 \times \Delta^1 \longrightarrow X \iff \Delta^1 \longrightarrow X^{\Delta^1}$$

It represents the desired arrow in $h(X^{\Delta^1})$

- Conservativity: omit it!

- $$qCat_2(A, X^{\Delta^1}) \cong h((X^{\Delta^1})^A)$$

$$\cong h((X^A)^{\Delta^1}) \longrightarrow (h(X^A))^{\mathbb{2}} = qCat_2(A, X)^{\mathbb{2}}$$

natural in A

- Put $A = X^{\Delta^1}$ and image of id will be:

$$\begin{tikzcd} & X^{\Delta^1} \arrow[dl, "d^1"'] \arrow[dr, "d^0"] & \\ X \arrow[rr, equal] \arrow[rr, phantom, shift left=4, "\overset{\iota}{\Rightarrow}"] & & X \end{tikzcd}$$

In general case:

a homotopy from f to g

$$A \longrightarrow X^{\Delta^1} \in \mathrm{Ob}\left(\mathrm{qCat}\left(A, X^{\Delta^1}\right)\right)$$

$$\rightsquigarrow$$

the composite 2-cell

$$\begin{tikzcd}
 & A \arrow[ddl, bend right, "f"'] \arrow[d] \arrow[ddr, bend left, "g"] & \\
 & X^{\Delta^1} \arrow[dl, "d^1"'] \arrow[dr, "d^0"] & \\
X \arrow[rr, equal] \arrow[rr, phantom, shift left=4, "\overset{\iota}{\Rightarrow}"] & & X
\end{tikzcd}$$

Surjectivity of $q\mathcal{C}at_2(A, X^{\Delta^1}) \longrightarrow (q\mathcal{C}at_2(A,X))^{\mathbb{2}}$ says:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & X \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} & A \arrow[d, "\Downarrow\tilde{\alpha}"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X^{\Delta^1} \arrow[dl, "d^1"'] \arrow[dr, "d^0"] \arrow[d, phantom, "\overset{\iota}{\Rightarrow}"] & \\ X \arrow[rr, equal] & & X \end{tikzcd}$$

$$\iota \cdot \tilde{\alpha} = \alpha$$

whiskering

- $\tilde{\alpha}$ is not unique: X^{Δ^1} is only a weak cotensor by $\mathbb{2}$
- The universal property defines the arrow quasi-categories up to equivalence

Lemma Let $\mathcal{Z}$ be in $qCat$, s.t. $\exists$ natural transformation

$$h(\mathcal{Z}^A) \longrightarrow \left(h(X^A)\right)^2$$

is smothering (i.e., surjective on objects, full and conservative)

Then

$$\mathcal{Z} \cong X^{\Delta^1}$$

▷ • $\exists$ $A = \mathcal{Z}$, the image of $1_{\mathcal{Z}} \in h(\mathcal{Z}^{\mathcal{Z}})$ is

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "e^1"'] \arrow[dr, "e^0"] \arrow[d, phantom, "\overset{\kappa}{\Longrightarrow}"] & \\ X \arrow[rr, equal] & {} & X \end{tikzcd}$$

• Surjectivity implies:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & A \arrow[d, "\breve{\alpha}"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & \mathcal{Z} \arrow[dl, "e^1"'] \arrow[dr, "e^0"] \arrow[d, phantom, "\Rightarrow \kappa"] & \\ X \arrow[rr, equal] & {} & X \end{tikzcd}$$

- Apply the weak univ. prop. of ι to k:

$$\tilde{k} : Z \longrightarrow X^{\Delta^1}$$

$$\begin{tikzcd}[row sep=small]
 & Z \arrow[d, "\tilde{k}"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & X^{\Delta^1} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\iota}{\Rightarrow}"] & \\
X \arrow[rr, equal] & & X
\end{tikzcd}$$

$$\iota \cdot \tilde{k} = k$$

- Apply the univ. prop. of k to ι:

$$\tilde{\iota} : X^{\Delta^1} \longrightarrow Z$$

$$\begin{tikzcd}[row sep=small]
 & X^{\Delta^1} \arrow[d, "\tilde{\iota}"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{k}{\Rightarrow}"] & \\
X \arrow[rr, equal] & & X
\end{tikzcd}$$

$$k \cdot \tilde{\iota} = \iota$$

- $X^{\Delta^1} \xrightarrow{\tilde{\iota}} Z \xrightarrow{\tilde{k}} X^{\Delta^1}$

gives a factorization ι through itself

$\Downarrow$

by conservativity & fullness

$$\mathcal{F}(\tilde{k}\tilde{\iota}) \cong \mathcal{F}(1_{X^{\Delta^1}}) \Longrightarrow \tilde{k}\tilde{\iota} \cong 1_{X^{\Delta^1}} \text{ in } h\left(\left(X^{\Delta^1}\right)^{X^{\Delta^1}}\right)$$

$$\mathcal{F} : h\left(\left(X^{\Delta^1}\right)^{X^{\Delta^1}}\right) \longrightarrow h\left(X^{\left(X^{\Delta^1}\right)}\right)^2$$

- Similarly,

$$i\tilde{k} \cong 1_{\mathcal{Z}} \quad \text{in } h(\mathcal{Z}^{\mathcal{Z}})$$

- These iso's are represented by

$$\mathbb{I} \longrightarrow (X^{\Delta^1})^{X^{\Delta^1}} \quad \& \quad \mathbb{I} \longrightarrow \mathcal{Z}^{\mathcal{Z}},$$

where $\mathbb{I} = N(\bullet \cong \bullet)$

and

Recall $f : \Delta^1 \to X$ is an iso in a quasi-category $\iff \exists$ an extension to $\mathbb{I}$

- So, we have

$$X^{\Delta^1} \times \Delta^1 \xrightarrow{\cong} X^{\Delta^1} \quad \text{by adjunction}$$

$$\mathcal{Z} \times \Delta^1 \xrightarrow{\cong} \mathcal{Z}$$

$\triangleleft$

Remark. It can be generalized to any category freely generated by a graph categories

Lemma $\forall$ diagram

$$\begin{tikzcd} E \arrow[r, "p"] & B & F \arrow[l, "q"'] \end{tikzcd}$$

a fibration in Joyal's model structure

cofibrant objects

of quasi-categories with q an isofibration

$$h\left(E \underset{B}{\times} F\right) \longrightarrow hE \underset{hB}{\times} hF$$

is smothering functor

Comma quasi-categories

$E \xrightarrow{p} B \xleftarrow{q} F$ is a diagram of quasi-categories

$$\begin{tikzcd} p\downarrow q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\Delta^1} \arrow[d] \\ E\times F \arrow[r, "p\times q"'] & B\times B \cong B^{\partial\Delta^1} \end{tikzcd}$$

an isofibration (i.e., fibration in Joyal's model structure)

Corollary The canonical functor

$$h(p\downarrow q) \longrightarrow \underbrace{h(E\times F)}_{h(B\times B)} \times h(B^{\Delta^1}) \longrightarrow \underbrace{(hE\times hF)}_{hB\times hB} \times (hB)^{2} = h(p)\downarrow h(q)$$

is smothering

this is the usual comma category in Cat

- The functor

$$h\left((p\downarrow q)^A\right) \longrightarrow h(E^A)\times h(F^A)\times \underset{h(B^A)\times h(B^A)}{h(B^A)^2} = h(p^A)\downarrow h(p^A)$$

is also smothering

- The weak universal property of $p\downarrow q$

$$\begin{tikzcd}
 & p\downarrow q \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
E \arrow[dr, "p"'] \arrow[rr, phantom, "\overset{\chi}{\Rightarrow}"] & & F \arrow[dl, "q"] \\
 & B &
\end{tikzcd}$$

— the image of the identity at $p\downarrow q$

Surjectivity gives

$$\begin{tikzcd}
 & A \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & \\
E \arrow[dr, "p"'] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & F \arrow[dl, "q"] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[d, dashed] \arrow[dl, bend right] \arrow[dr, bend left] & \\
E \arrow[dr, "g"'] & p\downarrow q \arrow[l] \arrow[r] & F \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

- By fullness & conservativity, one can derive that

whiskering

if $\quad \gamma \cdot (f, g : A \rightrightarrows p \downarrow q) = \alpha$

Then $\exists$ an iso

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \cong"] & p \downarrow q \end{tikzcd}$$

represented by a map $A \times \mathcal{I} \longrightarrow p \downarrow q$

- $\Longrightarrow$ $p \downarrow q$ is unique up to equivalence

The definition of ∞-cosmos

Def. An ∞-cosmos is a simplicially enriched category K:

- objects by def. are ∞-categories
- hom's are quasi-categories
- there are a subcategory of isofibrations $A \twoheadrightarrow B$

s.t. the following axioms hold:

(a) Completeness. – K posseses a terminal object 1;
cotensors A^U, $U \in sSet$;
pullbacks of isofibrations along any functor

(b) Isofibrations. – The class of isofibrations $\supset$ isomorphisms

$$\bigcup$$

$$! : A \twoheadrightarrow 1$$

- This class is stable under pullback along all functors
- $\exists\ p : E \twoheadrightarrow B,\ i : U \underset{\text{in sSet}}{\hookrightarrow} V \Rightarrow i \,\hat{\pitchfork}\, p : E^V \twoheadrightarrow E^U \times_{B^U} B^V$

the Leibnitz cotensor

$$\begin{tikzcd}
E^V \arrow[dr, dashed] \arrow[drr, bend left, "E^i"] \arrow[ddr, bend right, "f^V"'] & & \\
& \bullet \arrow[r] \arrow[d] & E^U \arrow[d, "f^U"] \\
& B^V \arrow[r, "B^i"] & B^U
\end{tikzcd}$$

- $\forall X \in Ob(K),\ \forall p : E \twoheadrightarrow B \Rightarrow fun(X, p) : fun(X, E) \twoheadrightarrow fun(X, B)$

(c) Equivalences. $f : A \xrightarrow{\sim} B$ is an equivalence when

$$fun(X, f) : fun(X, A) \longrightarrow fun(X, B)$$

is an equivalence of quasi-categories $\forall X \in K$

(d) Cofibrancy. All objects (say, A) are cofibrant

$$\begin{tikzcd}
& E \arrow[d, two heads, "\wr"] \\
A \arrow[r] \arrow[ur, dashed, "\exists"] & B
\end{tikzcd}$$

From the axioms above one can derive the stability of Fib^{tr}

(d) Trivial fibrations. — They are defined to be in $\mathcal{W}\mathcal{E} \cap Fib$

- They define a subcat $\supset$ Isomorphisms
- They are stable under pullback along all functors
- $i \pitchfork p : E^V \longrightarrow E^U \times_{B^U} B^V \in Fib^{tr}$ 　　$p : E \twoheadrightarrow B$, $i : U \hookrightarrow V$

When $p \in Fib^{tr}$ or $i \in CoFib^{tr}$ in Joyal model structure on sSet

Brown factorization lemma

(e) **Factorization.** $\forall$ functor $f : A \to B$ $\quad f = pj$

$$\begin{tikzcd} & N \arrow[dr, two heads, "p"] \arrow[dl, bend right, "q"', "\sim"] & \\ A \arrow[ur, "j"', "\sim"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

(f) **Cartesian closure.** $- \times - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ extends to

$$\mathrm{fun}(A \times B, C) \cong \mathrm{fun}(A, C^B) \cong \mathrm{fun}(B, C^A)$$

Examples

$\mathcal{M}_{cf}$ enriched over the Joyal model structure on sSet defines an ∞-cosmos

- Cat — the ∞-cosmos of small categories

  Isofib — the usual isofibrations of cats

  Equiv — the usual equivalences of cats

$$\begin{tikzcd} * \arrow[r] \arrow[d] & A \arrow[d] \\ \bullet \cong \bullet \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

- qCat — the ∞-cosmos of quasi-cat
- CSS — the ∞-cosmos complete Segal spaces
- Segal — the ∞-cosmos of Segal cats
- $sSet_+$ — the ∞-cosmos of naturally marked simplicial sets

- Θ_n-$\mathcal{S}p$ — the ∞-cosmos of Θ_n-spaces, a simplicial presheaf model of (∞, n)-categories

- $\mathcal{R}ezk_{\mathcal{M}}$ — the ∞-cosmos of Rezk objects in a nice model cat $\mathcal{M}$

They are used to define iterated Segal spaces — another simplicial presheaf model of (∞, n)-categories

Lemma. The equivalences in an ∞-cosmos are closed under retracts and satisfy 2-of-3 property

Lemma (trivial fibrations split)

The section defines a split fiber homotopy sequence

$$\begin{tikzcd} & E \arrow[d, "p", "\wr"'] \\ B \arrow[ur, dashed, "s"] \arrow[r, equal] & B \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[rr, bend left, "{(id_E, sp)}"] & E^{\mathbb{I}} \arrow[r, "{(ev_0, ev_1)}"'] \arrow[d, "p^{\mathbb{I}}"] & E \times E \\ B \arrow[r, "\Delta"'] & B^{\mathbb{I}} & \end{tikzcd}$$

and conversely $\forall$ isofibration that defines a split fiber homotopy equivalence is a trivial fibration

▷ If $p : E \overset{\sim}{\twoheadrightarrow} B$ is trivial $\Rightarrow$ $p_* : \mathrm{fun}(X, E) \overset{\sim}{\twoheadrightarrow} \mathrm{fun}(X, B)$

$\forall$ ∞-cat X

by the stability properties of $\mathcal{F}ib^{tr}$

$$
\begin{tikzcd}
\emptyset = \partial\Delta[0] \arrow[r] \arrow[d, hook] & \mathrm{fun}(B, E) \arrow[d, two heads, "\wr"', "p_*"] \\
\mathbb{1} = \Delta[0] \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dashed, "s"] & \mathrm{fun}(B, B)
\end{tikzcd}
$$

the trivial fibration

After that we can solve the problem

$$
\begin{tikzcd}
\mathbb{1} + \mathbb{1} \arrow[rr, "{(\mathrm{id}_E,\, sp)}"] \arrow[d, hook] & & \mathrm{fun}(E, E) \arrow[d, two heads, "\wr"] \\
\mathbb{I} \arrow[r, "!"'] \arrow[urr, dashed, "\alpha"] & \mathbb{1} \arrow[r, "p"'] & \mathrm{fun}(E, B)
\end{tikzcd}
$$

◁

Recall: Formal category theory in a 2-category

- Objects are called ∞-categories
- 1-cells $f : A \to B$ is said to be $(\infty\text{-})$functors
- 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ — $(\infty\text{-})$natural transformations

Def. An adjunction between ∞-categories:

- $f : B \to A$, $u : A \to B$
- $\eta : id_B \Rightarrow uf$ & $\varepsilon : fu \Rightarrow id_A$

$$\begin{tikzcd} & B \arrow[r, equal] \arrow[dr, "f"] & B \\ A \arrow[ur, "u"] \arrow[r, equal] & A \arrow[ur, "u"'] & \end{tikzcd} = u \left(\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Rightarrow id_u"] & B \end{tikzcd} \right) u \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[dr, "f"'] \arrow[dr, phantom, bend left, "\Downarrow\eta"] & B \arrow[dr, "f"] & \\ & A \arrow[u, "u"] \arrow[r, equal] \arrow[r, phantom, bend left, "\Downarrow\varepsilon"] & A \end{tikzcd} = f \left(\begin{tikzcd} B \arrow[d, bend left] \arrow[d, bend right] \arrow[d, phantom, "\Leftarrow id_f"] \\ A \end{tikzcd} \right) f$$

Prop. Adjunctions compose:

$$\begin{tikzcd} C \arrow[r, shift right, "u'"'] & B \arrow[l, shift right, "f'"'] \arrow[l, phantom, "\perp"] \arrow[r, shift right, "u"'] & A \arrow[l, shift right, "f"'] \arrow[l, phantom, "\perp"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[r, shift right, "u'u"'] & A \arrow[l, shift right, "ff'"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

Def. An equivalence between ∞-categories consists of:

- a pair of ∞-categories A and B
- $f : A \to B$ & $g : B \to A$ – functors
- $$\begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong \Downarrow \alpha"] & A \end{tikzcd} \quad \& \quad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\beta \Downarrow \cong"] & B \end{tikzcd}$$

The homotopy 2-category of an ∞-cosmos

Def. This is a strict 2-cat K_2 or hK so that

- $Ob(K_2) = Ob(K)$ — ∞-categories
- 1-cells $f : A \to B$ of K_2 $\leftrightsquigarrow$ the vertices $f \in \underbrace{\mathrm{Fun}(A,B)}_{\mathrm{qCat}}$

— ∞-functors

- 2 cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in K_2 $\leftrightsquigarrow$ homotopy classes of 1-simplicies

$$\alpha : f \longrightarrow g \in \mathrm{Fun}(A,B)$$

$$K_2 := h_* K \qquad \begin{tikzcd} \text{2-Cat} \arrow[r, phantom, "\perp"] & \text{sSet-Cat} \arrow[l, bend right, "h_*"'] \arrow[l, bend left] \end{tikzcd}$$

- The cat $qCat_2$ was first introduced by Joyal

Or, just $h\mathcal{K}$:

- $\mathrm{Ob}(h\mathcal{K}) := \mathrm{Ob}(\mathcal{K})$
- $\hom(A,B) := h(\mathrm{fun}(A,B))$

homotopy cat of quasi-cat

Def. The underlying cat of 2-cat — simply forgetting its 2-cells

$$u(\mathcal{K}) = u(h\mathcal{K})$$

Denote by $\mathbb{2} := N(\bullet \longrightarrow \bullet) = \Delta^1$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[l, bend left, "g"] \end{tikzcd} \quad \leftrightsquigarrow \quad \alpha : \mathbb{2} \longrightarrow \mathrm{fun}(A, B)$$

Transpose:

$$\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[r, "\alpha"] \arrow[dr, "g"'] & B^{\mathbb{2}} \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$ is an iso in $\mathcal{K}_2 \Longleftrightarrow \mathrm{ho}(\alpha) : \mathrm{ho}\,\mathbf{2} \longrightarrow \mathrm{ho}\,\mathrm{fun}(A, B)$ is iso

$\Longleftrightarrow \alpha : \mathbf{2} \longrightarrow \mathrm{fun}(A, B)$ extends to $\alpha' : \mathbb{I} \longrightarrow \mathrm{fun}(A, B)$

$$\mathbb{I} = N(\bullet \cong \bullet)$$

$$\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[r, "\alpha'"] \arrow[dr, "g"'] & B^{\mathbb{I}} \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & B \end{tikzcd}$$

Lemma. (i) $\forall$ 2-cell
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$
in $\mathcal{K}_2$ (the homotopy 2-cat of an ∞-cosmos)

is repr. by a map in qCat

$$\begin{tikzcd} \mathbb{1}+\mathbb{1} \arrow[d] \arrow[dr, "(f,g)"] & \\ \mathbb{2} \arrow[r, "\alpha"'] & \mathrm{fun}(A,B) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\ulcorner\alpha\urcorner"] \arrow[dr, "(g,f)"'] & B^{\mathbb{2}} \arrow[d, "(p_1,p_0)"] \\ & B\times B \end{tikzcd}$$

$\alpha \in$ qCat$_1$

(ii) $\forall$ invertible 2-cell
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & B \end{tikzcd}$$
in $\mathcal{K}_2$ —— // ——

$$\begin{tikzcd} \mathbb{1}+\mathbb{1} \arrow[d] \arrow[dr, "(f,g)"] & \\ \mathbb{I} \arrow[r, "\alpha"'] & \mathrm{fun}(A,B) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\ulcorner\alpha\urcorner"] \arrow[dr, "(g,f)"'] & B^{\mathbb{I}} \arrow[d, "(p_1,p_0)"] \\ & B\times B \end{tikzcd}$$

Functors between $h\mathcal{K}$

Lemma. $\forall$ simplicial functor $F : \mathcal{K} \longrightarrow \mathcal{L}$ induces a 2-functor $F : \mathcal{K}_2 \longrightarrow \mathcal{L}_2$

∞-cosmoi

▷ On objects and 1-cells — it's okay

$$
\begin{tikzcd}
\mathrm{fun}(A,B) \arrow[r, "F"] & \mathrm{fun}(FA,FB)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B & \longmapsto & FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB
\end{tikzcd}
$$

F is a morphism of simplicial sets

So, homotopic 1-cells map to homotopic ones ◁

Cartesian closure and products

Prop: (i) $\mathcal{K}_2$ has 2-categorical products

(ii) $\mathcal{K}_2$ is cartesian closed as 2-cat

__Theorem__ (Equivalences of ∞-categories are 2-cat equiv.)

In any ∞-cosmos $\mathcal{K}$ the F.A.E.:

(i) $\forall X \in \mathcal{K}$

$$f_* : \mathrm{fun}(X, A) \xrightarrow{\sim} \mathrm{fun}(X, B)$$

— equiv. of quasi-cat

(ii) $\exists\, g : B \to A$ & $\alpha : \mathrm{id}_A \cong gf$, $\beta : fg \cong \mathrm{id}_B$ in the homotopy 2-cat

(iii) $\exists\, g : B \to A$ and maps in the ∞-cosmos $\mathcal{K}$

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, two heads, "ev_0"] \arrow[d, two heads, "ev_1"'] \\ & A \end{tikzcd} \quad \& \quad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, two heads, "ev_0"] \arrow[d, two heads, "ev_1"'] \\ & B \end{tikzcd}$$

Thank you!