

The 2-category of quasi-categories

- The adjunction

$$h: \mathbf{sSet} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{Cat} : \mathcal{N}$$

- The counit is an isomorphism

Lemma. The functor $h: \mathbf{sSet} \rightarrow \mathbf{Cat}$ preserves finite product

▷ • $(h-)$ & $h(- \times -)$ preserve colimits

• \mathbf{sSet} & \mathbf{Cat} are cartesian closed

• $h\Delta^n \times h\Delta^m \stackrel{?}{\cong} h(\Delta^n \times \Delta^m)$

• $\Delta^n = \mathcal{N}(\tilde{n})$ where \tilde{n} is some category $h\mathcal{N} = \mathcal{E}$ is iso

• $(h\Delta^n) \times (h\Delta^m) \cong (h\mathcal{N}\tilde{n}) \times (h\mathcal{N}\tilde{m}) \cong \tilde{n} \times \tilde{m} \cong h\mathcal{N}(\tilde{n} \times \tilde{m}) \cong h(\mathcal{N}\tilde{n} \times \mathcal{N}\tilde{m})$ ▷

• Hence, \mathcal{h} and \mathcal{N} are strong monoidal

• $h_* : \mathcal{S}\text{Cat} \xrightleftharpoons{\perp} \mathcal{Q}\text{Cat} : \mathcal{N}_*$

Def. $q\text{Cat}_\infty \hookrightarrow \underline{\text{SSet}} \rightsquigarrow q\text{Cat}_2 := h_* q\text{Cat}_\infty$
2-category of quasi-categories

$$\text{Ob}(q\text{Cat}_2) = \{\text{quasi-categories}\}$$

$$1\text{-cells of } q\text{Cat}_2 = \{\text{maps of quasi-categories}\}$$

$$2\text{-cells of } q\text{Cat}_2 = \{\text{homotopy classes of homotopies}\}$$

$$2\text{-cell } \alpha : f \Rightarrow g \rightsquigarrow 1\text{-simplex } \tilde{\alpha} : f \rightarrow g \text{ in } \mathcal{Y}^X$$

$$\alpha_1 \sim \alpha_2 \iff \tilde{\alpha}_1 \sim \tilde{\alpha}_2 \text{ as } 1\text{-simplices in } \mathcal{Y}^X$$

Prop. $q\text{Cat}_2$ is cartesian closed

• $h: q\text{Cat} \rightarrow \text{Cat} \rightsquigarrow 2\text{-functor } h_{\downarrow}: q\text{Cat}_2 \rightarrow \underline{\text{Cat}}$

$$\begin{array}{ccc} h(Y^X) \times hX \cong h(Y^X \times X) & \xrightarrow{h(\text{ev})} & hY \\ & \downarrow \text{adj} & \\ & h(Y^X) & \rightarrow hY^{hX} \end{array} \quad \left. \vphantom{\begin{array}{ccc} h(Y^X) \times hX \cong h(Y^X \times X) & \xrightarrow{h(\text{ev})} & hY \\ & \downarrow \text{adj} & \\ & h(Y^X) & \rightarrow hY^{hX} \end{array}} \right\} \text{On hom-coats}$$

- The aim: the category theory of quasi-categories
- $q\text{Cat}_2$ has finite products (see above)
- 2-limits theory
- Cotensors with the walking arrow category \mathcal{I}

Weak limits in $q\text{Cat}_2$

$$\begin{aligned}
 \mathcal{F}: q\text{Cat}_2(A, X^{\mathcal{P}}) &\cong h((X^{\mathcal{P}})^A) \cong h((X^A)^{\mathcal{P}}) \stackrel{?}{\cong} (h(X^A))^{\mathcal{P}} \\
 &\cong (q\text{Cat}_2(A, X))^{\mathcal{P}}
 \end{aligned}$$

\swarrow cotensors commute with internal hom

It defines a notion of cotensoring by \mathcal{P}

- We require $h(X^{\mathcal{P}}) \cong (hX)^{\mathcal{P}} \longrightarrow \bullet$
- In $q\text{Cat}_\infty$ $X^{\Delta^1} \in q\text{Cat}$. Also, $\mathcal{N}\mathcal{P} = \Delta^1$ and $h\Delta^1 = \mathcal{P}$
 $h(X^{\Delta^1}) \longrightarrow (hX)^{h\Delta^1} \cong (hX)^{\mathcal{P}}$ is not iso

• A weak cotensor with \mathbb{Z}

Lemma (the universal property) The canonical comparison functor

$$h(X^{\Delta^1}) \rightarrow (hX)^{\mathbb{Z}}$$

is subjective on objects

full

&

conservative

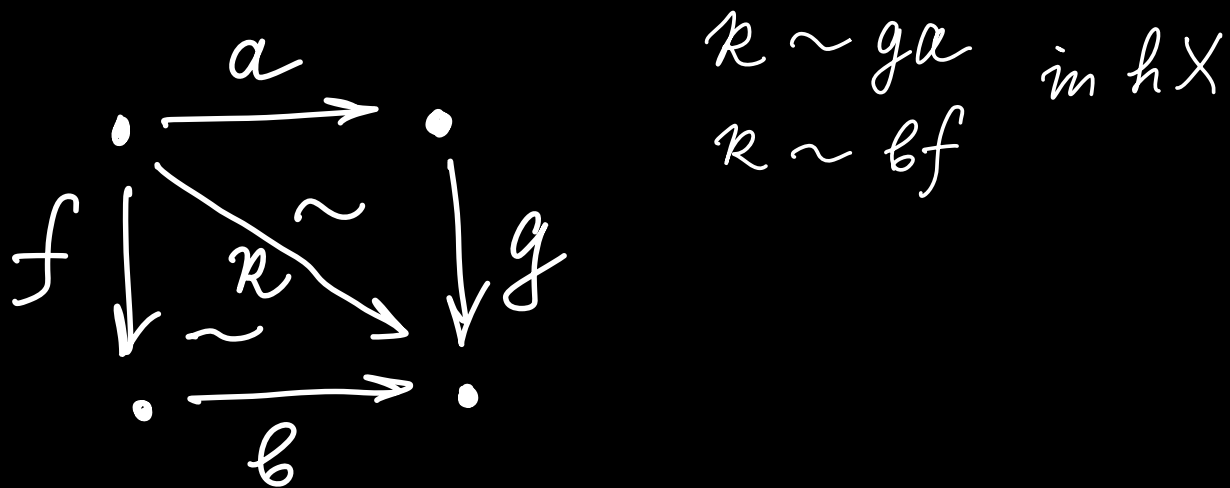
reflects iso

— it's smothering

▷ • Surjectivity: every arrow in hX is represented by a 1-simplex in X

• Fullness: find a morphism from $h(X^{\Delta^1})$ to

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ f \downarrow & & \downarrow g \\ \bullet & \xrightarrow{b} & \bullet \end{array}$$



We have

$$\Delta^1 \times \Delta^1 \longrightarrow X \iff \Delta^1 \longrightarrow X^{\Delta^1}$$

It represents the desired arrow in $h(X^{\Delta^1})$

- Conservativity: omit it!

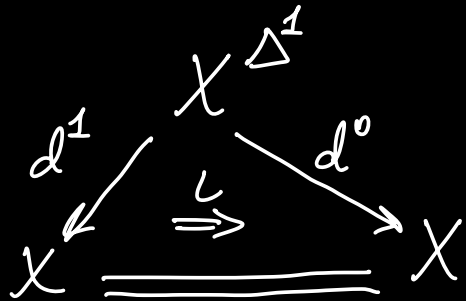


- $q\text{Cat}_2(A, X^{\Delta^1}) \cong h((X^{\Delta^1})^A)$

$$\cong h((X^A)^{\Delta^1}) \rightarrow (h(X^A))^{\mathcal{P}} = q\text{Cat}_2(A, X)^{\mathcal{P}}$$

natural in A

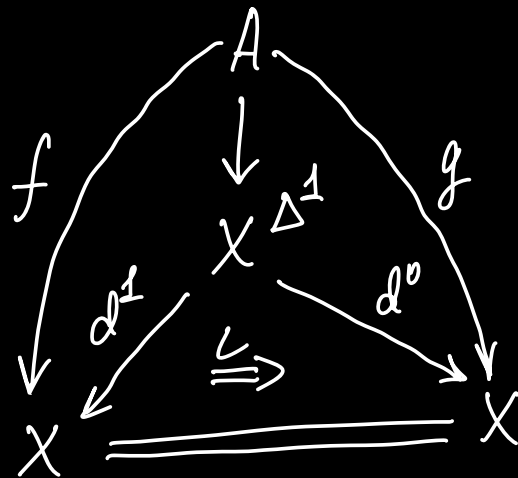
- Put $A = X^{\Delta^1}$ and image of id will be:



In general case:

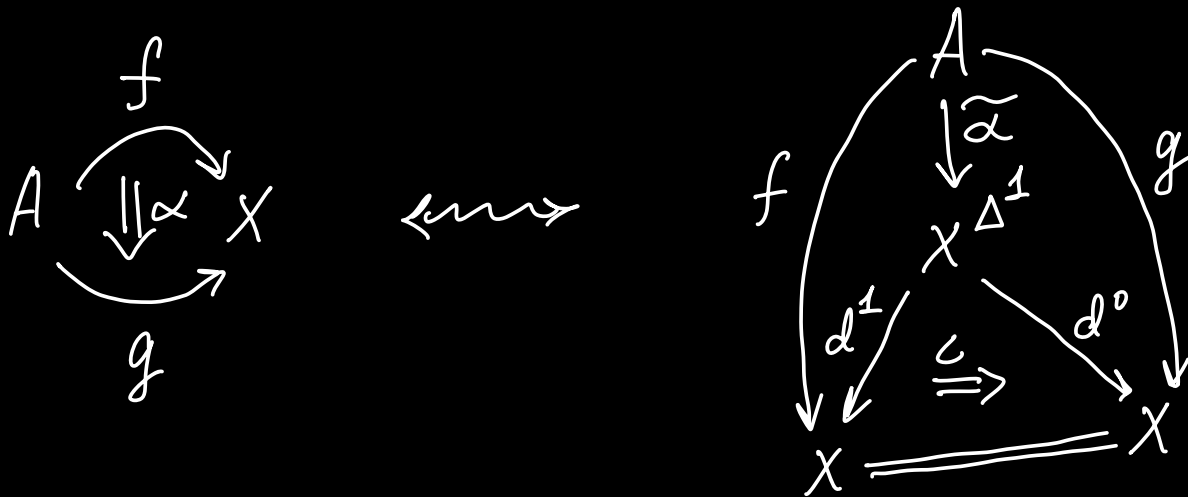
$$A \rightarrow X^{\Delta^1} \in \text{Ob}(\text{qCat}(A, X^{\Delta^1}))$$

a homotopy from f to g



the composite 2-cell

Surjectivity of $q\text{Cat}_2(A, X^{\Delta^1}) \rightarrow (q\text{Cat}_2(A, X))^{\mathcal{Z}}$ says:



$$\mathcal{L} \cdot \tilde{\alpha} = \alpha$$

whiskering

- $\tilde{\alpha}$ is not unique: X^{Δ^1} is only a weak cotensor by \mathcal{Z}
- The universal property defines the arrow quasi-categories up to equivalence

Lemma Let \mathcal{Z} be in $\mathcal{Q}Cat$, s.t. \exists natural transformation

$$h(\mathcal{Z}^A) \rightarrow (h(X^A))^{\mathcal{Z}}$$

is smothering (i.e., surjective on objects, full and conservative)

Then

$$\mathcal{Z} \cong X^{\Delta^1}$$

▷ • $\exists A = \mathcal{Z}$, the image of $1_{\mathcal{Z}} \in h(\mathcal{Z}^{\mathcal{Z}})$ is

$$\begin{array}{ccc} & \mathcal{Z} & \\ e^1 \swarrow & \mathcal{R} & \searrow e^0 \\ X & \xRightarrow{\quad} & X \end{array}$$

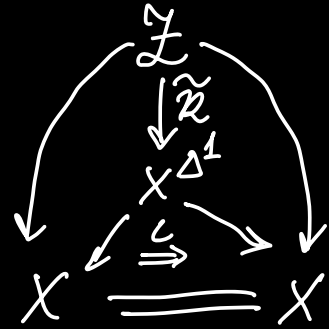
• Surjectivity implies:

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} X \rightsquigarrow$$

$$\begin{array}{ccc} & A & \\ & \downarrow \alpha & \\ & \mathcal{Z} & \\ e^1 \swarrow & \mathcal{R} & \searrow e^0 \\ X & \xRightarrow{\quad} & X \end{array}$$

- Apply the weak univ. prop. of \mathcal{L} to k :

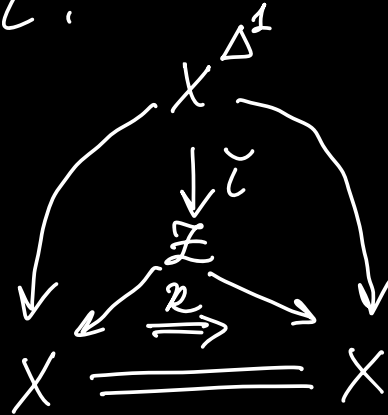
$$\tilde{\mathcal{R}}: \mathcal{Z} \rightarrow X^{\Delta^1}$$



$$L \cdot \tilde{\mathcal{R}} = \mathcal{R}$$

- Apply the univ. prop. of \mathcal{R} to L :

$$\tilde{\mathcal{L}}: X^{\Delta^1} \rightarrow \mathcal{Z}$$



$$\mathcal{R} \cdot \tilde{\mathcal{L}} = L$$

$$X^{\Delta^1} \xrightarrow{\tilde{\mathcal{L}}} \mathcal{Z} \xrightarrow{\tilde{\mathcal{R}}} X^{\Delta^1}$$

gives a factorization L through itself

$$\begin{aligned} &\Downarrow \\ &\mathcal{F}(\tilde{\mathcal{R}}\tilde{\mathcal{L}}) \cong \mathcal{F}(1_{X^{\Delta^1}}) \xRightarrow{\text{by conservativity \& fullness}} \tilde{\mathcal{R}}\tilde{\mathcal{L}} \cong 1_{X^{\Delta^1}} \text{ in } h((X^{\Delta^1})^{X^{\Delta^1}}) \\ &\mathcal{F}: h((X^{\Delta^1})^{X^{\Delta^1}}) \rightarrow h(X^{(X^{\Delta^1})})^{\mathcal{Z}} \end{aligned}$$

• Similarly,

$$\tilde{\mathcal{R}} \cong \mathbb{1}_{\mathcal{Z}} \text{ in } h(\mathcal{Z}^{\mathcal{Z}})$$

• These iso's are represented by

$$\mathcal{Y} \rightarrow (X^{\Delta^1})^{X^{\Delta^1}} \quad \& \quad \mathcal{Y} \rightarrow \mathcal{Z}^{\mathcal{Z}}$$

where $\mathcal{Y} = \mathcal{N}(\cdot \cong \cdot)$

and

Recall $f: \Delta^1 \rightarrow X$ is an iso in a quasi-category $\Leftrightarrow \exists$ an extension to \mathcal{Y}
to \mathcal{Y}

• So, we have $X^{\Delta^1} \times_{X^{\Delta^1}} X^{\Delta^1} \xrightarrow{\cong} X^{\Delta^1}$ by adjunction △

$$\mathcal{Z} \times_{\mathcal{Z}} \mathcal{Z} \xrightarrow{\cong} \mathcal{Z}$$

Remark It can be generalized to any category
freely generated by a graph categories

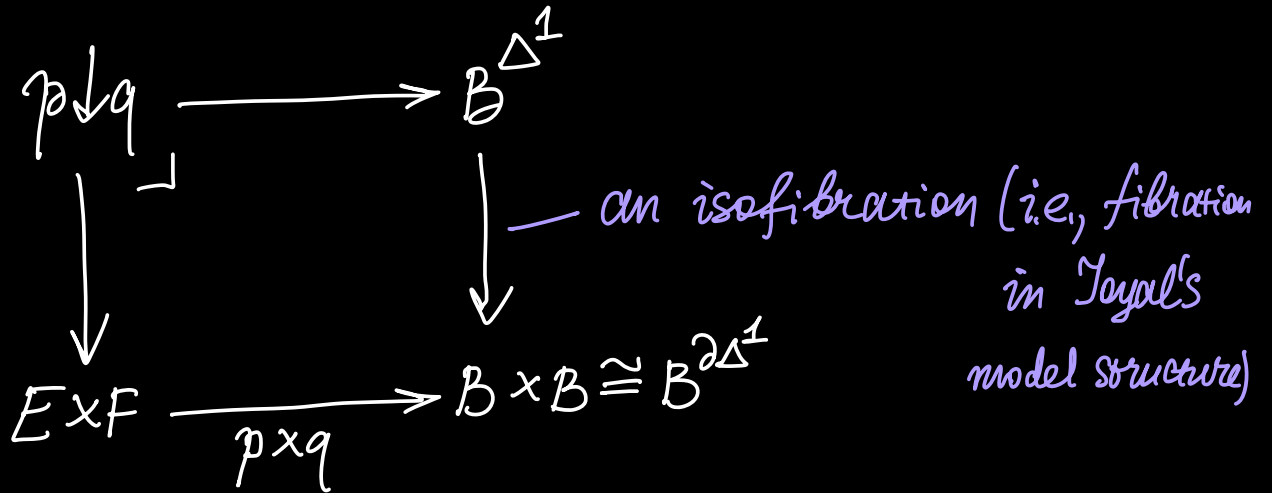
Lemma \forall diagram $E \xrightarrow{p} B \xleftarrow{q} F$ a fibration in Joyal's model structure
of quasi-categories with q an isofibration
cofibrant objects

$$h(E \times_B F) \longrightarrow hE \times_{hB} hF$$

is smothering functor.

Comma quasi-categories

$E \xrightarrow{p} B \xleftarrow{q} F$ is a diagram of quasi-categories



Corollary The canonical functor

$$h(p \downarrow q) \longrightarrow \underset{h(B \times B)}{h(E \times F) \times h(B^{\Delta^1})} \longrightarrow \underset{h(B \times B)}{(hE \times hF) \times (hB)^2} = h(p) \downarrow h(q)$$

is smothering

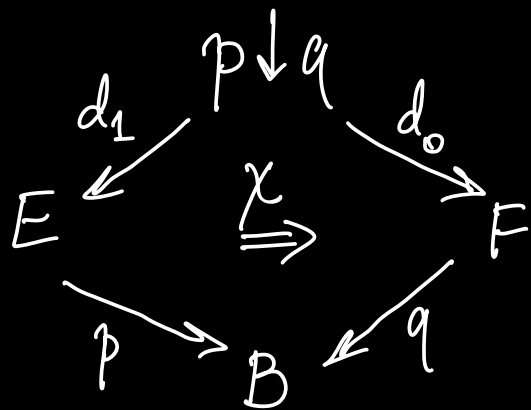
~~~~~  
 this is the usual comma category in  $\text{Cat}$

- The functor

$$h((p \downarrow q)^A) \rightarrow h(E^A) \times_{h(B^A) \times h(B^A)} h(F^A) \times h(B^A)^2 = h(p^A) \downarrow h(q^A)$$

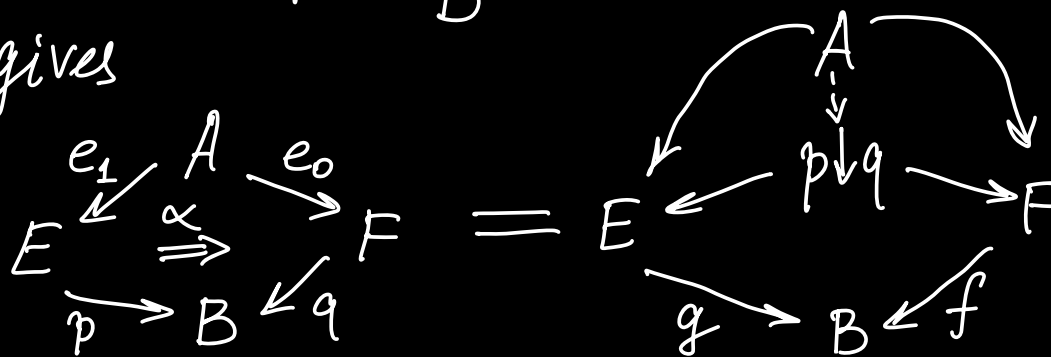
is also smothering

- The Weak universal property of  $p \downarrow q$



— the image of the identity at  $p \downarrow q$

Surjectivity gives



- By fullness & conservativity, one can derive that

$$\text{if } \gamma \cdot \overset{\text{whiskering}}{\left( f, g : A \rightrightarrows p \downarrow q \right)} = \alpha$$

Then  $\exists$  an iso

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \cong \\ \xrightarrow{g} \end{array} p \downarrow q$$

represented by a map  $A \times \mathcal{I} \rightarrow p \downarrow q$

- $\Rightarrow p \downarrow q$  is unique up to equivalence



## The definition of $\infty$ -cosmos

Def. An  $\infty$ -cosmos is a simplicially enriched category  $K$ :

- objects by def. are  $\infty$ -categories
- hom's are quasi-categories
- there are a subcategory of isofibrations  $A \twoheadrightarrow B$

s.t. the following axioms hold:

(a) Completeness. -  $K$  possesses a terminal object  $1$ ;  
cotensors  $A^U$ ,  $U \in \mathbf{Set}$ ;  
pullbacks of isofibrations along any functor

(b) Isofibrations. - The class of isofibrations  $\supset$  isomorphisms

$$! : A \overset{U}{\twoheadrightarrow} 1$$



From the axioms above one can derive the stability of  $\text{Fib}^{tz}$

(d) Trivial fibrations. — They are defined to be in  $\text{WF} \cap \text{Fib}$

— They define a subcat  $\supset$  Isomorphisms

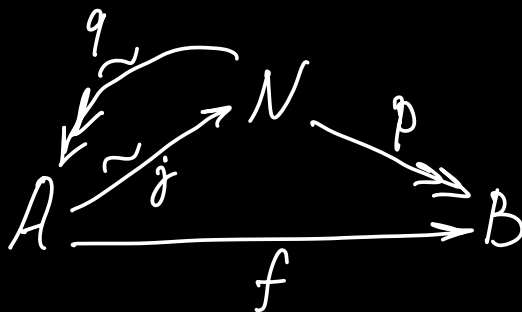
— They are stable under pullback along all functors

$$i \pitchfork p: E^V \longrightarrow E^U \times_{B^U} B^V \in \text{Fib}^{tz} \quad \begin{array}{l} p: E \twoheadrightarrow B \\ i: U \hookrightarrow V \end{array}$$

When  $p \in \text{Fib}^{tz}$  or  $i \in \text{Cofib}^{tz}$  in Joyal model structure on  $\text{sSet}$

Brown factorization lemma

(e) Factorization.  $\forall$  functor  $f: A \rightarrow B$   $f = p \circ j$



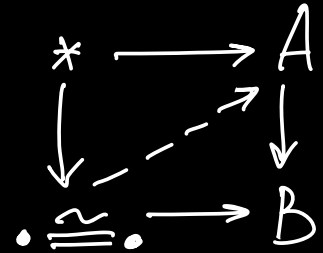
(f) Cartesian closure.  $- \times - : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$  extends to

$$\text{fun}(A \times B, C) \cong \text{fun}(A, C^B) \cong \text{fun}(B, C^A)$$

## Examples

$M_{cf}$  enriched over the Toyal model structure on  $sSet$  defines an  $\infty$ -cosmos

- $Cat$  — the  $\infty$ -cosmos of small categories
- $Isfib$  — the usual isofibrations of cats
- $Equiv$  — the usual equivalences of cats
- $qCat$  — the  $\infty$ -cosmos of quasi-cat
- $CSS$  — the  $\infty$ -cosmos complete Segal spaces
- $Segal$  — the  $\infty$ -cosmos of Segal cats
- $sSet_+$  — the  $\infty$ -cosmos of naturally marked simplicial sets



•  $\mathbb{H}_n\text{-Sp}$  — the  $\infty$ -cosmos of  $\mathbb{H}_n$ -spaces, a simplicial presheaf model of  $(\infty, n)$ -categories

•  $\text{Rezk}_{\mathcal{M}}$  — the  $\infty$ -cosmos of Rezk objects in a nice model cat  $\mathcal{M}$

They are used to define iterated Segal spaces — another simplicial presheaf model of  $(\infty, n)$ -categories

Lemma. The equivalences in an  $\infty$ -cosmos are closed under retraces and satisfy 2-of-3 property

Lemma (trivial fibrations split)

The section defines a split fiber homotopy sequence

$$\begin{array}{ccc} & & E \\ & \nearrow s & \downarrow p \\ B & \xlongequal{\quad} & B \end{array}$$

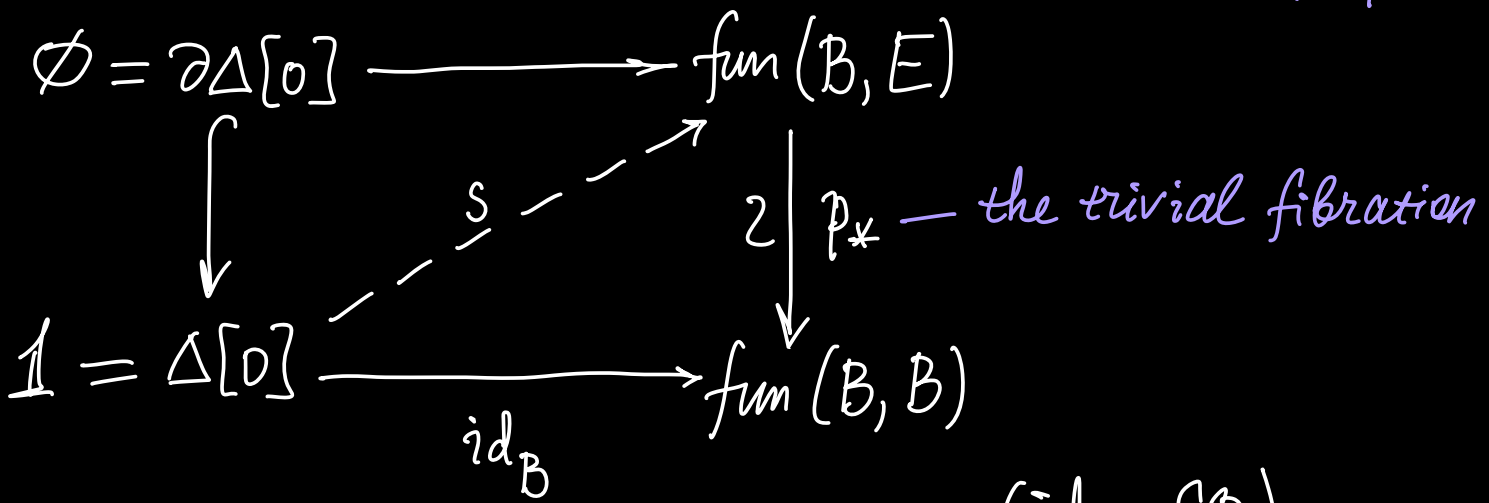
$$\begin{array}{ccccc} & & (id_E, sp) & & \\ & & \curvearrowright & & \\ E & \longrightarrow & E^{\text{II}} & \longrightarrow & \vec{E} \times E \\ p \downarrow & & \downarrow p^{\text{II}} & \swarrow (ev_0, ev_1) & \\ B & \xrightarrow{\quad \Delta \quad} & B^{\text{II}} & & \end{array}$$

and conversely,  $\forall$  isofibration that defines a split fiber homotopy equivalence is a trivial fibration

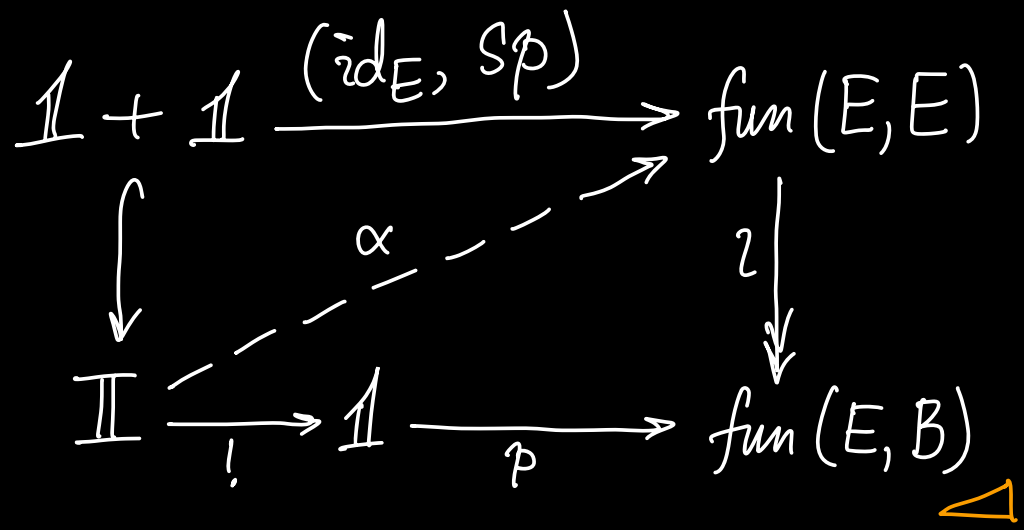
▷ If  $p: E \xrightarrow{\sim} B$  is trivial  $\Rightarrow p_*: \text{fun}(X, E) \xrightarrow{\sim} \text{fun}(X, B)$

$\forall \infty\text{-cat } X$

by the stability properties of  $\text{Fib}^{\text{tr}}$



After that we can solve the problem





## Recall: Formal category theory in a $\infty$ -category

- Objects are called  $\infty$ -categories
- 1-cells  $f: A \rightarrow B$  is said to be  $(\infty)$ -functors
- 2-cells  $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$  —  $(\infty)$ -natural transformations

Def. An adjunction between  $\infty$ -categories:

- $f: B \rightarrow A, u: A \rightarrow B$
- $\eta: id_B \Rightarrow uf$  &  $\epsilon: fu \Rightarrow id_A$

$$\begin{array}{c}
 B \xrightarrow{u} B \\
 \uparrow f \quad \downarrow u \\
 A \xrightarrow{u} A
 \end{array}
 = u \left( \begin{array}{c} B \\ \xrightarrow{id} B \\ \xRightarrow{idu} \\ A \end{array} \right) u
 \quad
 \begin{array}{c}
 B \xrightarrow{u} B \\
 \downarrow f \quad \uparrow u \\
 A \xrightarrow{u} A
 \end{array}
 = f \left( \begin{array}{c} B \\ \xrightarrow{id} B \\ \xRightarrow{idf} \\ A \end{array} \right) f$$

Prop. Adjunctions compose:

$$C \begin{array}{c} \xleftarrow{f'} \\ \perp \\ \xrightarrow{u'} \end{array} B \begin{array}{c} \xleftarrow{f} \\ \perp \\ \xrightarrow{u} \end{array} A \rightsquigarrow C \begin{array}{c} \xleftarrow{ff'} \\ \perp \\ \xrightarrow{u'u} \end{array} A$$

Def. An equivalence between  $\infty$ -categories consists of:

- a pair of  $\infty$ -categories  $A$  and  $B$
- $f: A \rightarrow B$  &  $g: B \rightarrow A$  — functors

• 
$$A \begin{array}{c} \xrightarrow{f} \\ \cong \\ \downarrow \alpha \\ \xrightarrow{g} \end{array} A \quad \& \quad B \begin{array}{c} \xrightarrow{fg} \\ \cong \\ \downarrow \beta \\ \xrightarrow{gf} \end{array} B$$

# The homotopy 2-category of an $\infty$ -cosmos

Def. This is a strict 2-cat  $K_2$  or  $hK$  so that

- $Ob(K_2) = Ob(K) = \infty\text{-categories}$
  - 1-cells  $f: A \rightarrow B$  of  $K_2 \iff$  the vertices  $f \in \text{fun}(A, B)$
  - $\infty$ -functors  $f$
  - 2 cells  $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$  in  $K_2 \iff$  homotopy classes of 1-simplices
- $\alpha: f \longrightarrow g \in \text{fun}(A, B)$
- $K_2 := h_* K$
- $2\text{-Cat} \xleftarrow{h_*} \text{Set-Cat}$
- The cat  $qCat_2$  was first introduced by Joyal

Or, just  $h\mathcal{K}$ :

$$\bullet \text{ob}(h\mathcal{K}) := \text{ob}(\mathcal{K})$$

$$\bullet \text{hom}(A, B) := h(\text{fun}(A, B))$$

homotopy cat of quasi-cat

Def. The underlying cat of 2-cat — simply forgetting  
its 2-cells

$$u(\mathcal{K}) = u(h\mathcal{K})$$

Denote by  $\mathcal{Q} := \mathcal{N}(\cdot \longrightarrow \cdot) = \Delta^1$

$$\begin{array}{c}
 f \\
 \curvearrowright \\
 A \quad \Downarrow \alpha \quad B \\
 \curvearrowleft \\
 g
 \end{array}
 \iff \alpha : \mathcal{Q} \longrightarrow \text{fun}(A, B)$$

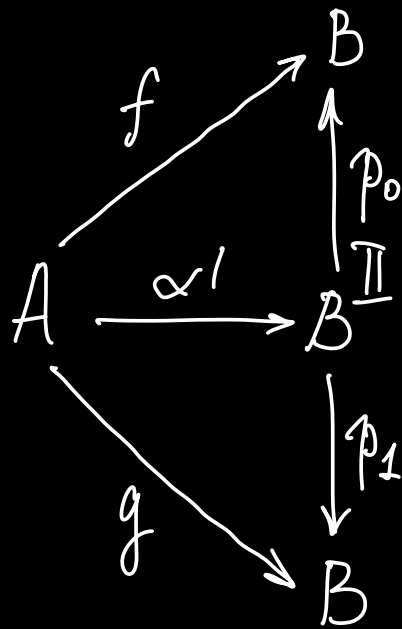
Transpose:

$$\begin{array}{ccc}
 & & B \\
 & \nearrow f & \\
 A & \xrightarrow{\alpha} & B^{\mathcal{Q}} \\
 & \searrow g & \\
 & & B \\
 & & \downarrow p_1 \\
 & & B \\
 & & \uparrow p_0
 \end{array}$$

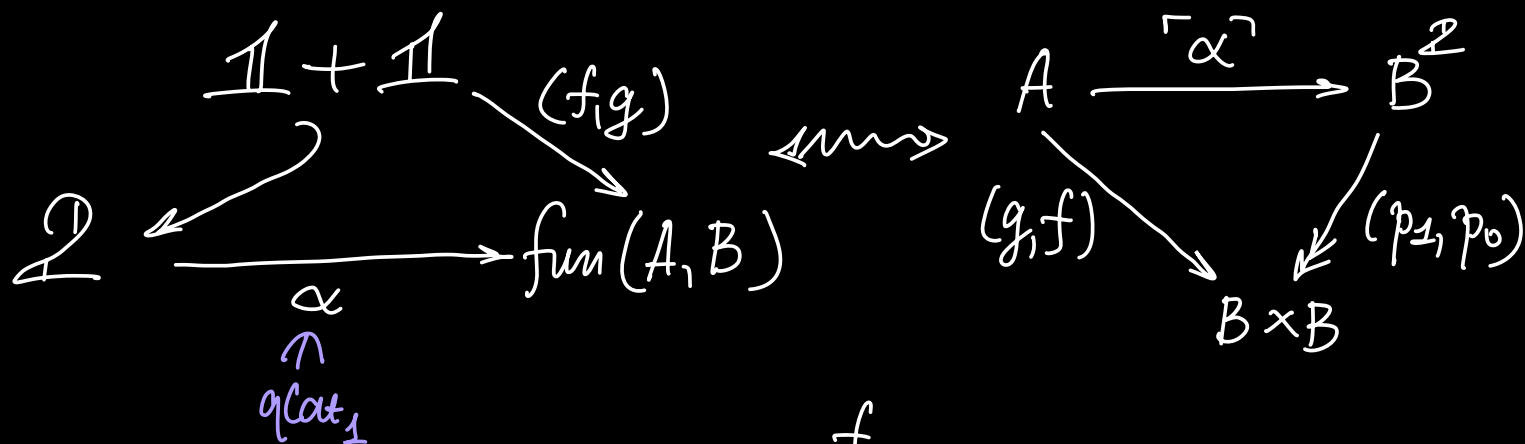
$$\begin{array}{c}
 f \\
 \curvearrowright \\
 A \xrightarrow{\quad} B \\
 \Downarrow \alpha \\
 \curvearrowleft \\
 g
 \end{array}
 \text{ is an iso in } \mathcal{K}_2 \iff \text{ho}(\alpha): \text{ho} \mathcal{I} \rightarrow \text{ho fun}(A, B) \text{ is iso}$$

$$\iff \alpha: \mathcal{I} \rightarrow \text{fun}(A, B) \text{ extends to } \alpha': \mathbb{I} \rightarrow \text{fun}(A, B)$$

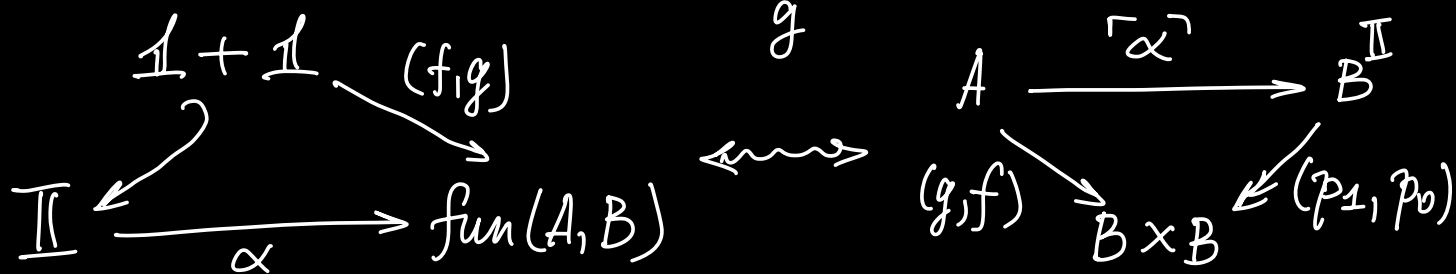
$$\mathcal{N}(\cdot \cong \cdot)$$



Lemma. (i)  $\forall$  2-cell  $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$  in  $\mathcal{K}_2$   
 is repr. by a map in  $qcat$  the homotopy 2-cat of an  $\infty$ -cosmos



(ii)  $\forall$  invertible 2-cell  $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \cong \alpha \\ \xrightarrow{g} \end{array} B$  in  $\mathcal{K}_2$  — " —



## Functors between $h\mathcal{K}$

Lemma.  $\forall$  simplicial functor  $F: \mathcal{K} \rightarrow \mathcal{L}$   <sup>$\infty$ -cosmoi</sup> induces  
a  $\mathcal{L}$ -functor,  $F: \mathcal{K}_2 \rightarrow \mathcal{L}_2$

▷ On objects and 1-cells — it's okay

$$\begin{array}{ccc}
 \text{fun}(A, B) & \xrightarrow{F} & \text{fun}(FA, FB) \\
 \begin{array}{ccc}
 \xrightarrow{f} & & \\
 \Downarrow \alpha & & \\
 \xrightarrow{g} & & 
 \end{array} & \longmapsto & \begin{array}{ccc}
 \xrightarrow{Ff} & & \\
 \Downarrow F\alpha & & \\
 \xrightarrow{Fg} & & 
 \end{array}
 \end{array}$$

$F$  is a morphism of simplicial sets

So, homotopic 1-cells map to homotopic ones





## Cartesian closure and products

- Prop.
- (i)  $\mathcal{K}_2$  has 2-categorical products
  - (ii)  $\mathcal{K}_2$  is cartesian closed as 2-cat

Theorem (Equivalences of  $\infty$ -categories are 2-cat equiv.)

In any  $\infty$ -cosmos  $\mathcal{K}$  the F.A.E.:

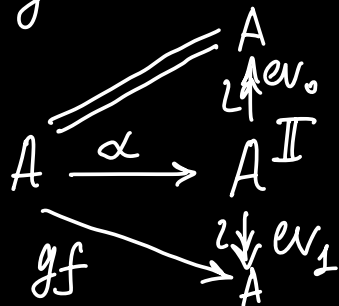
(i)  $\forall X \in \mathcal{K}$

$$f_x: \text{fun}(X, A) \xrightarrow{\sim} \text{fun}(X, B)$$

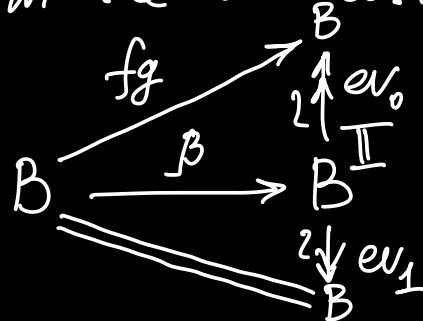
— equiv. of quasi-cat

(ii)  $\exists g: B \rightarrow A$  &  $\alpha: \text{id}_A \cong gf$ ,  $\beta: fg \cong \text{id}_B$  in the homotopy 2-cat

(iii)  $\exists g: B \rightarrow A$  and maps in the  $\infty$ -cosmos  $\mathcal{K}$



&



Thank you!