

Two-Valued Groups, Chazy Equation, Dubrovin–Frobenius Structures, and QYBE

Mikhail Kornev

HSE University, Steklov Mathematical Institute

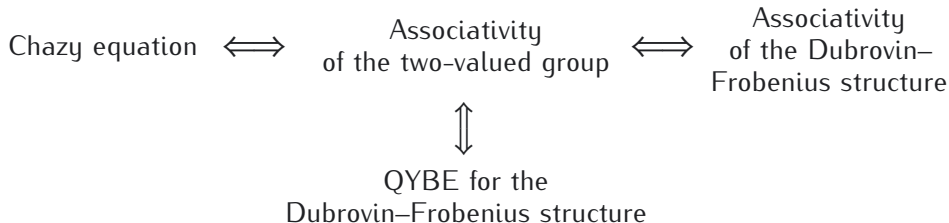
*Integrable Systems and Quantum Theory
(Pogrebkov-80)*

October 8, 2026 Saint Petersburg

Joint work with Victor Buchstaber and Vladimir Rubtsov [BKR26]

One associative law in different contexts

Main result:



The four displayed conditions reduce to $4k_8 = k_4^2 - k_2k_6$.

n -valued groups

An n -valued product is a map

$$* : X \times X \longrightarrow \mathrm{Sym}^n(X), \quad x * y = [u_1, \dots, u_n],$$

with repetitions retained. Products of multisets concatenate all outputs.

Definition

An n -valued group has an associative product, an identity $e \in X$, and an inverse map $\mathrm{inv} : X \rightarrow X$ satisfying

$$\begin{aligned} (x * y) * z &= x * (y * z) \quad \text{in } \mathrm{Sym}^{n^2}(X), \\ x * e &= e * x = \underbrace{[x, \dots, x]}_{n \text{ entries}}, \\ e &\in x * \mathrm{inv}(x), \quad e \in \mathrm{inv}(x) * x. \end{aligned}$$

In the algebraic setting X is a variety and the maps may be rational. The group is *regular* when both maps are defined everywhere.

The finite-automorphism coset construction

Let G be a group and $A \subset \text{Aut}(G)$ a finite subgroup, $|A| = n$. Write $X = G/A$ for the orbit space and $\pi : G \rightarrow X$.

Coset construction

$$\pi(g) * \pi(h) = [\pi(g \alpha(h))]_{\alpha \in A}.$$

$$e_X = \pi(e_G), \quad \text{inv}(\pi(g)) = \pi(g^{-1}).$$

Changing the representatives g, h only permutes the n entries.

- Coincident outputs are retained, including at points with stabilizers.

[Buc06]

An elementary two-valued group

For $x, y \in \mathbb{C}$, define the two-valued product

$$x * y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2].$$

Changing either square root exchanges the two entries.

The quadratic equation

$$p_2(z; x, y) = z^2 - 2(x + y)z + (x - y)^2 = 0.$$

$$0 * x = [x, x], \quad x * x = [0, 4x], \quad \text{inv}(x) = x.$$

The n -valued groups $\mathbb{G}_n$

Let ε be a primitive n th root of unity. On $\mathbb{C}$, put

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n \mid r = 0, \dots, n-1].$$

This multiset is independent of the choices of the roots. The neutral element is 0, and $\text{inv}(x) = (-1)^n x$.

The Buchstaber–Rees polynomials

With $a^n = x$ and $b^n = y$, the symmetric polynomial is

$$p_n(z; x, y) = \prod_{r=0}^{n-1} (z - (-1)^n (a + \varepsilon^r b)^n).$$

The outputs of $x * y$ are the roots of $p_n(z; (-1)^n x, (-1)^n y)$.

For algebraically independent x, y , $p_n(z; (-1)^n x, (-1)^n y)$ is the minimal polynomial of $(a + b)^n$ over $\mathbb{C}(x, y)$, of degree n .

Wendt determinant formulas

The polynomial $p_n(z; x, y)$ is the determinant of the Wendt (x, y, z) -matrix. In each display, $w^n = z$.

$$p_2 = \begin{vmatrix} w^2 - x + y & 2wy \\ 2w & w^2 - x + y \end{vmatrix},$$

$$p_3 = \begin{vmatrix} w^3 + x + y & 3yw & 3yw^2 \\ 3w^2 & w^3 + x + y & 3yw \\ 3w & 3w^2 & w^3 + x + y \end{vmatrix},$$

$$p_4 = \begin{vmatrix} w^4 - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & w^4 - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & w^4 - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & w^4 - x + y \end{vmatrix}.$$

Symmetric two-algebraic two-valued laws

Consider a polynomial:

$$P_2(z; x, y) = F_0(x, y)z^2 + F_1(x, y)z + F_2(x, y).$$

Definition

A *symmetric 2-algebraic two-valued group* on $\mathbb{C}$ is an algebraic two-valued group with neutral element 0 and inverse map $\text{inv}(x) = x$, whose multiplication is defined by

$$x * y = [z \mid P_2(z; x, y) = 0].$$

The polynomial P_2 is symmetric in the three variables x, y, z and has degree at most two in each variable.

The universal associativity condition

Put $e_1 = x + y + z$, $e_2 = xy + xz + yz$, and $e_3 = xyz$. The universal symmetric 2-algebraic law is

$$P_2(z; x, y) = e_1^2 - 4e_2 + k_2e_3 + k_4e_1e_3 + k_6e_2e_3 + k_8e_3^2.$$

Its two roots in z define $x * y$. In particular,

$$P_2(z; x, 0) = (z - x)^2, \quad P_2(0; x, x) = 0.$$

Theorem (Buchstaber–Kornev–Rubtsov–Veselov)

$$(x * y) * z = x * (y * z) \iff 4k_8 - k_4^2 + k_2k_6 = 0.$$

The universal coefficient ring is

$$\mathbb{Z}[k_2, k_4, k_6, k_8]/(4k_8 - k_4^2 + k_2k_6).$$

The universal coefficient ring

The coefficient ring of the universal family is

$$R_0 = \mathbb{Z}[k_2, k_4, k_6, k_8]/(4k_8 - k_4^2 + k_2k_6).$$

Universal property (Buchstaber–Kornev–Rubtsov, 2026)

The family P_2 is initial in the class of polynomials defining symmetric 2-algebraic two-valued groups over $\mathbb{Z}[1/2]$ -algebras.

For any such polynomial Q with coefficient ring R' , there is a unique homomorphism

$$r : R_0 \longrightarrow R', \quad Q = r(P_2).$$

Over $\mathbb{Z}[1/2]$ -algebras, the universal family has independent parameters k_2, k_4, k_6 , and

$$k_8 = \frac{k_4^2 - k_2k_6}{4}.$$

Elliptic curves appear unexpectedly

The associativity condition for the symmetric polynomial leads to the cubic

$$E_a : \quad \eta^2 = \zeta^3 + a_1\zeta^2 + a_2\zeta + a_3.$$

For $\delta_a \neq 0$, this is an elliptic curve.

Corollary (Buchstaber–Kornev, 2026)

In the reciprocal coordinate $x = 1/\zeta$, the elliptic quotient law is

$$\mathcal{B}_a(z; x, y) = (e_1 - a_2 e_3)^2 - 4(1 + a_3 e_3)(e_2 + a_1 e_3) = 0,$$

with

$$k_2 = -4a_1, \quad k_4 = -2a_2, \quad k_6 = -4a_3, \quad k_8 = a_2^2 - 4a_1a_3.$$

Elliptic curves emerge from the algebraic associativity condition.

[Buc90]

The elliptic coset construction

Let

$$E : \quad \eta^2 = \zeta^3 + a_1\zeta^2 + a_2\zeta + a_3, \quad \delta_a = \text{Disc}_t(t^3 + a_1t^2 + a_2t + a_3).$$

Proposition (Buchstaber–Gaiur–Rubtsov, 2026)

The universal symmetric 2-algebraic two-valued group is obtained by the coset construction for the involution

$$(\zeta, \eta) \longmapsto (\zeta, -\eta).$$

Its multiplication is given by the Buchstaber polynomial:

$$\pi(P) * \pi(Q) = [\pi(P + Q), \pi(P - Q)].$$

If $\delta_a \neq 0$, the multiplication is defined everywhere on $\mathbb{CP}^1 \times \mathbb{CP}^1$. If $\delta_a = 0$, it is defined on a Zariski-dense domain, using the smooth points of the degenerate cubic.

[BGR26]

A family containing all elliptic isomorphism classes

Use $D = (2\pi i)^{-1} d/d\tau$. The Ramanujan identities are

$$DE_2 = \frac{E_2^2 - E_4}{12}, \quad DE_4 = \frac{E_2E_4 - E_6}{3}, \quad DE_6 = \frac{E_2E_6 - E_4^2}{2}.$$

Proposition (Buchstaber–Kornev–Rubtsov, 2026)

$$\eta^2 = \zeta^3 + a_1(\tau)\zeta^2 + a_2(\tau)\zeta + a_3(\tau),$$

$$a_1 = -\pi^2 E_2, \quad a_2 = 4\pi^4 DE_2, \quad a_3 = -\frac{8\pi^6}{3} D^2 E_2.$$

As $\tau \in \mathfrak{H}$ varies, all smooth elliptic classes occur.

- The group-law coefficients are $k_2 = -4a_1$, $k_4 = -2a_2$, $k_6 = -4a_3$.

The coefficient flow of Buchstaber polynomials

Let

$$B_\tau(z; x, y) = e_1^2 - 4e_2 + k_2(\tau)e_3 + k_4(\tau)e_1e_3 + k_6(\tau)e_2e_3 \\ + \frac{k_4(\tau)^2 - k_6(\tau)k_2(\tau)}{4}e_3^2$$

be a one-parameter family of Buchstaber polynomials satisfying

$$D = \frac{1}{2\pi i} \frac{d}{d\tau}, \quad \lambda \neq 0, \quad Dk_2 = \frac{\lambda}{2}k_4, \quad Dk_4 = \frac{3\lambda}{4}k_6, \\ Dk_6 = \frac{\lambda}{4}(k_4^2 - k_2k_6) = \lambda k_8.$$

All elliptic isomorphism classes in one family

For the displayed family of Buchstaber polynomials and coefficient flow:

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

There exists a one-parameter family $G_{\mathbb{C}}(B_{\tau})$ of symmetric 2-algebraic two-valued groups containing representatives of all coset isomorphism classes of symmetric 2-algebraic two-valued groups corresponding to coset constructions of elliptic curves

$$y^2 = x^3 - \frac{k_2}{4}x^2 - \frac{k_4}{2}x - \frac{k_6}{4}, \quad \delta_k \neq 0.$$

Here δ_k is the discriminant of the cubic on the right.

Associativity is equivalent to the Chazy equation

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

Within the family $G_{\mathbb{C}}(B_{\tau})$, the associativity condition of the universal symmetric 2-algebraic two-valued group

$$4k_8 = k_4^2 - k_6k_2$$

is equivalent to the Chazy equation

$$D^3y = yD^2y - \frac{3}{2}(Dy)^2,$$

where

$$y(\tau) = -\frac{\lambda}{4}k_2(\tau), \quad D = \frac{1}{2\pi i} \frac{d}{d\tau}.$$

The elliptic family above has $\lambda = -1/\pi^2$ and $y = E_2$.

The nodal and cuspidal families

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

The nodal cubic corresponds to the constant family $k_2 \equiv 1$, $k_4 \equiv k_6 \equiv 0$.

The cuspidal cubic corresponds to the family $k_2 = k_4 = k_6 = 0$.

Nodal case: the Chebyshev coset monoid

$$\tau(u) = u^{-1}, \quad \pi(u) = \frac{1}{2}(u + u^{-1}).$$

$$s * t = [st \pm \sqrt{(s^2 - 1)(t^2 - 1)}], \quad e = 1.$$

Domain: $\text{Sym}^2(\mathbb{CP}^1) \setminus \{[\infty, \infty]\}$.

With $s = 1 + 2x$, a rescaled nodal representative is

$$\eta^2 = \zeta^2(\zeta + 1), \quad a_1 = 1, \quad a_2 = a_3 = 0.$$

Cuspidal representative

$$\eta^2 = \zeta^3.$$

The additive coset group $\mathbb{G}_2$:

$$x * y = [(\sqrt{x} \pm \sqrt{y})^2].$$

Its quotient coordinate is

$$x = v^2, \text{ where } v = \zeta/\eta.$$

The Möbius action on Chazy solutions

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \quad g\tau = \frac{a\tau + b}{c\tau + d}.$$

For the Möbius action use the ordinary derivative. Put $Y = \pi i y$, where y solves the normalized Chazy equation; then

$$Y''' = 2YY'' - 3(Y')^2, \quad (-)' = \frac{d}{d\tau}.$$

The Möbius action

$$(Y \mid_2 g)(\tau) = \frac{Y(g\tau)}{(c\tau + d)^2} - \frac{6c}{c\tau + d}.$$

In the normalized- D coordinate, the same action reads [Cha11]:

$$(y \mid_2 g)(\tau) = \frac{y(g\tau)}{(c\tau + d)^2} - \frac{6c}{\pi i(c\tau + d)}.$$

Three Möbius orbits

Let Sol be the solution space of the Chazy equation, with its normalized derivative $D = (2\pi i)^{-1} d/d\tau$.

Theorem (Chazy, 1911)

The space Sol decomposes into three orbits

$$\text{Sol} = G(E_2) \sqcup G(0) \sqcup G(1), \quad G = \text{SL}_2(\mathbb{C}).$$

They are the orbit of the Eisenstein series $y_0 = E_2$, the zero solution $y_1 = 0$, and the constant solution $y_2 = 1$.

[Cha11, Oh95, BLP03]

Classification of the flow families

Corollary (Buchstaber–Kornev–Rubtsov, 2026)

Any one-parameter family $G_{\mathbb{C}}(B_{\tau})$ of symmetric 2-algebraic two-valued groups defined by the coefficient flow is obtained from one of the three flow families above by the action of an element of $G = \mathrm{SL}_2(\mathbb{C})$ under the Möbius action.

Corollary (Buchstaber–Kornev–Rubtsov, 2026)

The classes of G -equivalence of one-parameter families $G_{\mathbb{C}}(B_{\tau})$ are in one-to-one correspondence with

$$G(E_2), \quad G(0), \quad G(1)$$

of the solutions $y = E_2$, $y = 0$, and $y = 1$ of the Chazy equation.

Frobenius algebras

Definition

A *Frobenius algebra* over $\mathbb{C}$ is unital associative algebra A equipped with a linear functional $\theta : A \rightarrow \mathbb{C}$ for which the bilinear form

$$\eta_A(a, b) = \theta(ab)$$

is nondegenerate:

$$\eta_A(a, b) = 0 \text{ for every } b \in A \implies a = 0.$$

Any Frobenius algebra is automatically finite-dimensional.

The form is invariant under multiplication:

$$\eta_A(ab, c) = \eta_A(a, bc) \quad (a, b, c \in A).$$

The coalgebra of a Frobenius algebra

Proposition (Farnsteiner, 2007; Kock, 2003)

Every Frobenius algebra (A, θ) carries a unique Frobenius-compatible coalgebra structure (A, Δ, θ) with counit θ . If

$$\Delta(1) = \sum_i a_i \otimes b_i,$$

then

$$\Delta(x) = \sum_i a_i x \otimes b_i = \sum_i a_i \otimes x b_i.$$

For dual bases $(x_i), (y_i)$, with $\langle x_i, y_j \rangle = \delta_{ij}$,

$$\Delta(x) = \sum_i x_i x \otimes y_i = \sum_i x_i \otimes x y_i.$$

Dubrovin–Frobenius manifolds

One can define Frobenius manifolds locally using *potential functions* in flat coordinates $t^1, \dots, t^n$. We recall the data needed below.

A smooth (or holomorphic) potential $\mathcal{F}(t^1, \dots, t^n)$ and a constant, nondegenerate symmetric metric $\eta = (\eta_{\alpha\beta})$ determine the tangent multiplication by

$$\eta(\partial_\alpha \circ \partial_\beta, \partial_\gamma) = \partial_\alpha \partial_\beta \partial_\gamma \mathcal{F}, \quad \partial_\alpha = \frac{\partial}{\partial t^\alpha}.$$

The unit condition for $e = \partial_1$ is

$$\partial_1 \partial_\alpha \partial_\beta \mathcal{F} = \eta_{\alpha\beta}.$$

Associativity is equivalent to the *WDVV equation*.

When WDVV holds, each tangent space $T_p M$ is a commutative Frobenius algebra with this multiplication and the invariant pairing η_p .

[Dub96]

WDVV and multiplication matrices

Let $\mathcal{F}(t^1, \dots, t^n)$ be the *Frobenius potential*, $n = \dim M$. The constant Gram matrix $\eta_{\alpha\beta}$ has inverse $\eta^{\alpha\beta}$. In flat coordinates put

$$\mathcal{F}_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma \mathcal{F}, \quad (C_\alpha)^\gamma{}_\beta = \eta^{\gamma\delta} \mathcal{F}_{\alpha\beta\delta}.$$

Thus C_α is multiplication by ∂_α . Repeated Greek indices are summed.

The WDVV equation

$$\mathcal{F}_{\alpha\beta\mu} \eta^{\mu\nu} \mathcal{F}_{\nu\gamma\delta} = \mathcal{F}_{\alpha\gamma\mu} \eta^{\mu\nu} \mathcal{F}_{\nu\beta\delta}.$$

$$\text{WDVV} \quad \Longleftrightarrow \quad [C_\alpha, C_\beta] = 0 \quad \text{for all } \alpha, \beta.$$

A tangent multiplication from a potential

In flat coordinates $u = t^1$, $x = t^2$, $y = t^3$, take

$$\mathcal{F}(u, x, y) = \frac{1}{2}u^2y + \frac{1}{2}ux^2 + f(x, y), \quad \eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The unit is $e_1 = \partial_u$. The third derivatives define

$$c_{\alpha\beta}^{\gamma} = \eta^{\gamma\nu} \mathcal{F}_{\alpha\beta\nu}, \quad a = f_{xxy}, \quad b = f_{xxx}, \quad c = f_{xyy}, \quad d = f_{yyy}.$$

Multiplication table

$$\begin{aligned} e_2^2 &= ae_1 + be_2 + e_3, \\ e_2e_3 &= ce_1 + ae_2, \\ e_3^2 &= de_1 + ce_2. \end{aligned}$$

Associativity is the scalar WDVV equation

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

The multiplication table is associative precisely when

$$\kappa = a^2 - bc - d = 0 \quad \Longleftrightarrow \quad f_{xxy}^2 = f_{xxx}f_{xyy} + f_{yyy}.$$

Pointwise, the corresponding two-valued law has coefficients

$$(k_2, k_4, k_6, k_8) = 4(b, a, c, d),$$

$$4k_8 - k_4^2 + k_2k_6 = -16\kappa.$$

The two-valued law and the tangent algebra have the same associativity condition.

The associativity ideal

Take the multiplication table of the three-dimensional potential over

$$K = \mathbb{Z}[a, b, c, d].$$

Proposition (Buchstaber–Kornev–Rubtsov, 2026)

The minimal ideal $I \subset K$ for which the table becomes associative over K/I is

$$I = (a^2 - d - bc).$$

With $x = t^2$, $y = t^3$, this is the scalar equation

$$f_{xxy}^2 - f_{yyy} - f_{xxx}f_{xyy} = 0.$$

Multiplication matrices and WDVV

The matrices C_α introduced above specialize in this basis to

$$e_\alpha = \partial_{t^\alpha}, \quad C_1 = I_3, \quad C_2 = M = L_{e_2}, \quad C_3 = N = L_{e_3}.$$

Thus M and N are the same multiplication matrices, with

$$M = L_{e_2} = \begin{pmatrix} 0 & a & c \\ 1 & b & a \\ 0 & 1 & 0 \end{pmatrix}, \quad N = L_{e_3} = \begin{pmatrix} 0 & c & d \\ 0 & a & c \\ 1 & 0 & 0 \end{pmatrix}.$$

Their commutator is

$$[M, N] = (a^2 - bc - d) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\text{WDVV} \iff a^2 - bc - d = 0 \iff MN = NM.$$

The Dubrovin algebra as a cubic quotient

For each point of the associative domain, the tangent Frobenius algebra is

$$T_p M \simeq \mathbb{C}[t]/(t^3 - bt^2 - 2at - c).$$

Here t is a polynomial indeterminate.

Proposition

$$\varphi(e_1) = 1, \quad \varphi(e_2) = t, \quad \varphi(e_3) = t^2 - bt - a.$$

Two-valued associativity and the Chazy potential

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

The condition $4k_8 = k_4^2 - k_6k_2$ in the universal symmetric 2-algebraic two-valued group is equivalent to associativity of the Dubrovin–Frobenius structure on a three-dimensional domain with potential

$$F(t) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{96} \varphi(t^3).$$

For this potential, the scalar associativity equation becomes

$$\varphi''' = \varphi\varphi'' - \frac{3}{2}(\varphi')^2, \quad (-)' = \frac{d}{dt^3}.$$

The coproduct in the three-dimensional basis

The dual bases are (e_1, e_2, e_3) and (e_3, e_2, e_1) . The Frobenius functional is

$$\theta(e_1) = \theta(e_2) = 0, \quad \theta(e_3) = 1.$$

On the associative locus,

$$\Delta(e_1) = e_1 \otimes e_3 + e_2 \otimes e_2 + e_3 \otimes e_1,$$

$$\begin{aligned} \Delta(e_2) = & ce_1 \otimes e_1 + a(e_1 \otimes e_2 + e_2 \otimes e_1) \\ & + be_2 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_2, \end{aligned}$$

$$\Delta(e_3) = de_1 \otimes e_1 + c(e_1 \otimes e_2 + e_2 \otimes e_1) + ae_2 \otimes e_2 + e_3 \otimes e_3.$$

$\Delta(e_1)$ is the Casimir tensor.

The WDVV tensor train and QYBE

Fix $p \in M$ and put $V = T_p M$. Under WDVV the matrices C_α commute. For arbitrary constants $\lambda^{\alpha\beta}$, form the finite tensor train

$$S = \sum_{\alpha, \beta} \lambda^{\alpha\beta} C_\alpha \otimes C_\beta.$$

On $V^{\otimes 3}$ put $S_{12} = S \otimes I$, $S_{23} = I \otimes S$, and $S_{13} = T_{23}(S \otimes I)T_{23}$.

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}.$$

All three tensor-leg operators commute, so their product is independent of order. The inverse metric gives the canonical choice

$$Q = \sum_{\alpha, \beta} \eta^{\alpha\beta} C_\alpha \otimes C_\beta = I \otimes N + M \otimes M + N \otimes I.$$

The Frobenius algebra, its Casimir element, and QYBE

Let (A, θ) be a finite-dimensional Frobenius algebra over $\mathbb{C}$, with dual bases satisfying $\theta(e_i e^j) = \delta_i^j$. For the left regular representation $L_a(x) = ax$, put

$$Q = \sum_i L_{e_i} \otimes L_{e^i} \in \text{End}(A \otimes A).$$

Equivalently, $Q(x \otimes y) = \sum_i e_i x \otimes e^i y$.

Theorem (Beidar–Fong–Stolin, 1997)

For every such Frobenius algebra, including the noncommutative case,

$$Q_{12} Q_{23} Q_{12} = Q_{23} Q_{12} Q_{23}.$$

With $T(x \otimes y) = y \otimes x$, the operator $R = QT$ satisfies

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

The explicit $R(a, b, c, d)$ matrix

Use the ordered basis $(e_1 \otimes e_1, e_1 \otimes e_2, \dots, e_3 \otimes e_3)$.

$$R = \begin{pmatrix} 0 & c & d & c & a^2 & ac & d & ac & c^2 \\ 0 & a & c & a & ab+c & bc+d & c & a^2 & ac \\ 1 & 0 & 0 & 0 & a & c & 0 & c & d \\ 0 & a & c & a & ab+c & a^2 & c & bc+d & ac \\ 1 & b & a & b & 2a+b^2 & ab+c & a & ab+c & a^2 \\ 0 & 1 & 0 & 1 & b & a & 0 & a & c \\ 1 & 0 & 0 & 0 & a & c & 0 & c & d \\ 0 & 1 & 0 & 1 & b & a & 0 & a & c \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Here $a = f_{xxy}$, $b = f_{xx}$, $c = f_{xyy}$, $d = f_{yyy}$.

Writing $\kappa = a^2 - bc - d$, the rank is $\text{rank } R = 3$ for $\kappa = 0$, and 7 for $\kappa \neq 0$.

QYBE for the four-parameter matrix

Theorem (Buchstaber–Kornev–Rubtsov, 2026)

The displayed matrix $R(a, b, c, d)$ satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \iff a^2 - d - bc = 0.$$

For the multiplication matrices $M = L_{e_2}$ and $N = L_{e_3}$,

$$R = (I_3 \otimes N + M \otimes M + N \otimes I_3) T,$$

where $T(e_i \otimes e_j) = e_j \otimes e_i$ is the flip.

Two-valued associativity and QYBE

For the Dubrovin–Frobenius structure with the Chazy potential

$$F(t) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{96} \varphi(t^3),$$

and its Casimir operator

$$Q = L_{e_1} \otimes L_{e_3} + L_{e_2} \otimes L_{e_2} + L_{e_3} \otimes L_{e_1}.$$

Corollary (Buchstaber–Kornev–Rubtsov, 2026)

The universal symmetric 2-algebraic two-valued associativity condition

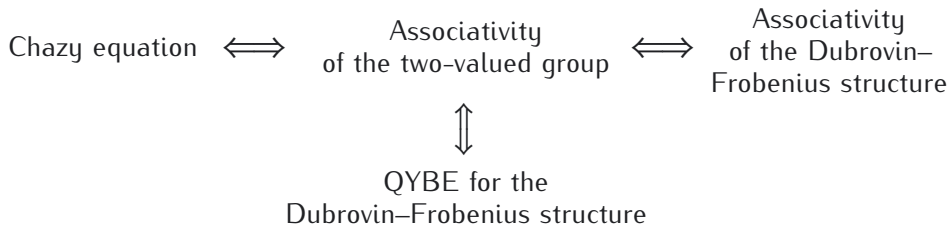
$$4k_8 = k_4^2 - k_6 k_2$$

is equivalent to QYBE for this Dubrovin–Frobenius Casimir construction.

The equivalence diagram

Main result

$$4k_8 = k_4^2 - k_2k_6$$



References

- [BKR26] V. Buchstaber, M. Kornev, V. Rubtsov, *Two-Valued Groups, Chazy Equation, Dubrovin–Frobenius Structures, and QYBE*, arXiv:2604.26641 (2026), preprint.
- [BK26] V. M. Buchstaber, M. I. Kornev, *Algebraic n -valued groups, cubics, and Wendt matrices*, Mat. Sb. **217**(8) (2026), 29–86.
- [BGR26] V. Buchstaber, I. Gaiur, V. Rubtsov, *Algebraic 2-Valued Group Structures on $\mathbb{P}^1$, Kontsevich-type Polynomials, and Multiplication Formulas, I*, Izvestiya: Math. **90**(1) (2026), 34–69.
- [Buc06] V. M. Buchstaber, *n -valued groups: theory and applications*, Mosc. Math. J. **6**(1) (2006), 57–84.
- [Buc75] V. M. Buchstaber, *Two-valued formal groups. Algebraic theory and applications to cobordism. I*, Math. USSR-Izv. **9**(5) (1975), 987–1006.
- [Buc90] V. M. Buchstaber, *Functional equations associated with addition theorems for elliptic functions and two-valued algebraic groups*, Russian Math. Surveys **45**(3) (1990), 213–215.

References (continued)

- [Cha11] J. Chazy,
Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Acta Math. **34** (1911), 317–385.
- [Oh95] Y. Ohyama,
Differential relations of theta functions, Osaka J. Math. **32**(2) (1995), 431–450.
- [BLP03] V. M. Buchstaber, D. V. Leikin, M. V. Pavlov,
Egorov Hydrodynamic Chains, the Chazy Equation, and $SL(2, \mathbb{C})$, Funct. Anal. Appl. **37**(4) (2003), 251–262.
- [Dub96] B. Dubrovin,
Geometry of 2D topological field theories, Lecture Notes in Math. **1620**, Springer (1996), 120–348.
- [Koc03] J. Kock,
Frobenius Algebras and 2-D Topological Quantum Field Theories, Cambridge University Press (2003).
- [BV19] V. M. Buchstaber, A. P. Veselov,
Conway Topograph, $PGL_2(\mathbb{Z})$ -Dynamics and Two-Valued Groups, Russian Math. Surveys **74**(3) (2019), 387–430.
- [Far07] R. Farnsteiner,
Self-Injective Algebras: Frobenius Algebras and Coalgebras, lecture notes (2007).
- [BFS97] K. I. Beidar, Y. Fong, A. Stolin,
On Frobenius Algebras and the Quantum Yang–Baxter Equation, Trans. Amer. Math. Soc. **349**(9) (1997), 3823–3836.

Thank you for your attention!