

Buchstaber, Ochanine, Krichever, and Witten Genera

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Automorphic Forms and Applications

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Based on

1. M. Kornev, *Buchstaber, Ochanine, Krichever, and Witten Genera*, arXiv:2603.21118.
2. V. Buchstaber and M. Kornev, *Logarithm of the Universal Two-Valued Formal Group*, arXiv:2609.15878.

From characteristic series to genera

Let R be a commutative $\mathbb{Q}$ -algebra and choose a series

$$Q(z) = 1 + q_1 z + q_2 z^2 + \cdots \in R[[z]].$$

For a closed stably complex manifold M^{2n} with tangent Chern roots $x_1, \dots, x_r$, its Hirzebruch genus is

$$\varphi_Q(M) = \left\langle \prod_{j=1}^r Q(x_j), [M] \right\rangle.$$

- ▶ Take the component of cohomological degree $2n$ before evaluating.
- ▶ Disjoint union and Cartesian product give addition and multiplication: $\varphi_Q: \Omega_U \longrightarrow R$ is a ring homomorphism.

The same genus is encoded by the strict series $f(z) = z/Q(z)$, called its exponential.

The geometric meaning of logarithm and exponential

Let $g = f^{-1}$ under composition. The associated formal group is

$$F(u, v) = f(g(u) + g(v)), \quad g(F(u, v)) = g(u) + g(v).$$

Two geometric families recover all coefficients:

$$g(u) = u + \sum_{n \geq 1} \varphi(\mathbb{CP}^n) \frac{u^{n+1}}{n+1},$$

$$f(u) = u + \sum_{n \geq 1} \varphi(\Theta_n) \frac{u^{n+1}}{(n+1)!}.$$

Here Θ_n is a smooth theta divisor in a general principally polarized abelian variety of dimension $n+1$.

Inverting a formal series relates projective spaces to theta divisors; the different denominators carry arithmetic information.

One-dimensional n -valued formal groups

Buchstaber and Novikov introduced two-valued formal groups in connection with complex and symplectic cobordism, as well as the **modulus-square construction** from one-valued to two-valued formal groups. [BN71]

Definition

Let R be a commutative ring. A one-dimensional n -valued formal group is encoded by a monic polynomial

$$P(Z; x, y) = Z^n - \theta_0(x, y)Z^{n-1} + \cdots + (-1)^n \theta_{n-1}(x, y) \in R[[x, y]][[Z]].$$

Its roots define the unordered product

$$x * y = [z_1, \dots, z_n], \quad P(z_j; x, y) = 0,$$

which is associative, has a unit 0 with $x * 0 = 0 * x = [x, \dots, x]$, and admits an inverse $\text{inv}(x)$ with $0 \in x * \text{inv}(x)$ and $0 \in \text{inv}(x) * x$.

Buchstaber's two-valued classification

Theorem (Buchstaber, 1975)

Let R be an integral-domain $\mathbb{Q}$ -algebra. Every commutative two-valued formal group is strongly isomorphic either to the elementary cyclic law

$$\mathbb{G}_2 : \quad x * y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2],$$

or to the diagonal law

$$x * y = [x + y, x + y].$$

If z_1, z_2 are the roots of $P(Z; x, y)$, a strict series $B(x) = x + O(x^2)$ gives a strong isomorphism $P \rightsquigarrow Q$ when

$$\prod_{j=1}^2 (Z - B(z_j)) = Q(Z; B(x), B(y)).$$

For $Q = \mathbb{G}_2$, the series B is called the **logarithm**; Buchstaber found it explicitly.

From associativity to an additive coordinate

For a one-dimensional formal group law over a $\mathbb{Q}$ -algebra, **associativity** and a two-sided unit give an additive coordinate.

The invariant differential and logarithm

Put

$$\omega(u) = \left. \frac{\partial F(u, v)}{\partial v} \right|_{v=0}, \quad g(u) = \int_0^u \frac{dt}{\omega(t)}.$$

Differentiating associativity at the unit gives

$$\omega(F(u, v)) = \frac{\partial F(u, v)}{\partial v} \omega(v), \quad g(F(u, v)) = g(u) + g(v).$$

Hence commutativity follows from associativity and the unit.

Abel assumed the stronger condition that $F(z, F(x, y))$ is symmetric in x, y, z . [Abel26, Haz78]

An n -valued analogue of Abel's theorem

Assume associativity and a **strong two-sided unit**. In the formal setting,
$$P(Z; x, 0) = (Z - x)^n, \quad P(Z; 0, y) = (Z - y)^n.$$

Theorem (Kornev, 2026)

Over a reduced commutative $\mathbb{Q}$ -algebra, every one-dimensional formal n -valued multiplication with these properties is commutative:

$$P(Z; x, y) = P(Z; y, x).$$

Theorem (Kornev, 2026)

Over a field of characteristic zero, every associative algebraic n -valued monoid on $\mathbb{P}^1$ with a strong two-sided unit is commutative.

No inverse axiom is required. Over an arbitrary commutative $\mathbb{Q}$ -algebra, the coefficients of $P(Z; x, y) - P(Z; y, x)$ lie in the nilradical.

[Kor, in prep.]

Complete classification by divisors of n

Let R be an integral-domain $\mathbb{Q}$ -algebra and P an associative one-dimensional formal n -valued multiplication with a strong two-sided unit.

Theorem (Kornev, 2026)

There is a **unique divisor** $m \mid n$ and a strong coordinate $B(t) = t + O(t^2) \in R[[t]]$ such that, for the roots z_v of P ,

$$\prod_{v=1}^n (W - B(z_v)) = p_m(W; B(x), B(y))^{n/m}.$$

Here p_m is the polynomial of the elementary cyclic law $\mathbb{G}_m$:

$$p_m(W; X, Y) = \prod_{j=0}^{m-1} \left(W - \left(\sqrt[m]{X} + \zeta_m^j \sqrt[m]{Y} \right)^m \right).$$

The divisor m is the generator's differential order and the number of distinct generic values; each value occurs with multiplicity n/m .

$$\text{Disc}_7 P \neq 0 \iff m = n.$$

Where the two values come from

Along $\mathbb{CP}^\infty \rightarrow \mathbb{HP}^\infty$, a quaternionic line restricts to $\eta \oplus \bar{\eta}$. For two such lines, the complex tensor product splits as

$$(\eta_1 \oplus \bar{\eta}_1) \otimes (\eta_2 \oplus \bar{\eta}_2) = \xi_+ \oplus \xi_-,$$

$$\xi_+ = \eta_1 \eta_2 \oplus \bar{\eta}_1 \bar{\eta}_2, \quad \xi_- = \eta_1 \bar{\eta}_2 \oplus \bar{\eta}_1 \eta_2.$$

Each $\xi_\pm$ is a quaternionic line of the form $L \oplus \bar{L}$. Writing $u = c_1^U(\eta_1)$ and $v = c_1^U(\eta_2)$ gives

$$x = u\bar{u}, \quad y = v\bar{v},$$

$$z_+ = F(u, v)F(\bar{u}, \bar{v}), \quad z_- = F(u, \bar{v})F(\bar{u}, v).$$

The symmetric functions $z_+ + z_-$ and $z_+ z_-$ lie in $\Omega_U[[x, y]]$.

[BK26, BN71]

Weak units and polynomial pullbacks

Example (Kornev)

The multiplication

$$x * y = [x + y, x - y]$$

defines an associative two-valued algebraic multiplication on $\mathbb{A}^1$. Since $x * 0 = [x, x]$ and $0 * x = [x, -x]$, the element 0 is a weak unit, but not a strong unit.

weak unit: $x \in 0 * x$ and $x \in x * 0$

strong unit: $0 * x = x * 0 = [x, \dots, x]$

Proposition (Kornev, 2026)

Let $P(Z; x, y) = 0$ represent an algebraic n -valued monoid on $\mathbb{A}_{\mathbb{C}}^1$ with strong unit 0. Let $T \in \mathbb{C}[z]$ be monic of degree $m \geq 2$ with $T(0) = 0$. Then

$$P_T(Z; x, y) := P(T(Z); T(x), T(y)) = 0$$

defines an associative algebraic mn -valued monoid with weak unit 0. If P is commutative, then so is P_T .

Example: the Fermat curve. Pulling back the additive algebraic group by $T(z) = z^n$ gives

$$P_{\mathbb{F}_n}(Z; x, y) = Z^n - x^n - y^n, \quad x *_{\mathbb{F}_n} y = \left[\zeta_n^k \sqrt[n]{x^n + y^n} \right]_{k=0}^{n-1}.$$

Thus $0 * y = y * 0 = [\zeta_n^k y]_{k=0}^{n-1}$, so 0 is weak but not strong for $n > 1$.

Buchstaber's universal two-valued family

In 1990 Buchstaber introduced

$$\mathcal{B}_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz), \quad \mathbf{a} = (a_1, a_2, a_3).$$

Each $\mathcal{B}_a$ defines a two-valued formal group and is universal among modulus squares of the formal groups

$$\mathcal{F}(u, v) = \frac{u^2\lambda_1(v) - v^2\lambda_1(u)}{u\lambda_2(v) - v\lambda_2(u)}.$$

[Buc90]

Buchstaber–Krichever family $\mathcal{F} \xrightarrow{\text{modulus square}} \mathcal{B}_a \xleftarrow{\text{modulus square}} \text{formal group } F_{\text{Bc}}$

The modulus-square construction is not injective: distinct one-valued formal groups can determine the same two-valued law.

Theorem (Kornev, 2026): the formal group F_{Bc}

Put

$$R_a(t) = 1 - a_1 t^2 + a_2 t^4 - a_3 t^6.$$

► Then

$$F_{Bc}(u, v) = \left[\left(\frac{u\sqrt{R_a(v)} - v\sqrt{R_a(u)}}{u^2 - v^2} \right)^2 + a_3 u^2 v^2 \right]^{-1/2}$$

is a one-dimensional formal group law with formal inverse $\bar{u} = -u$.

- Its modulus square is exactly the two-valued law $\mathcal{B}_a(z; x, y)$.
- Its exponential is odd, a fact used below for the values on theta divisors.

Theorem (Kornev, 2026): logarithms and exponentials

► Define

$$g_2 = 4 \left(\frac{a_1^2}{3} - a_2 \right), \quad g_3 = 4 \left(\frac{a_1 a_2}{3} - \frac{2a_1^3}{27} - a_3 \right).$$

Two-valued law $\mathcal{B}_a$

$$B(x) = \left(\int_0^{\sqrt{x}} \frac{dt}{\sqrt{1 + a_1 t^2 + a_2 t^4 + a_3 t^6}} \right)^2 \quad B^{-1}(x) = \frac{1}{\wp(\sqrt{x}; g_2, g_3) - a_1/3}.$$

One-valued law F_{Bc}

$$g_{\text{Bc}}(u) = \int_0^u \frac{dt}{\sqrt{1 - a_1 t^2 + a_2 t^4 - a_3 t^6}} \quad f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3) + a_1/3}}.$$

For singular Weierstrass cubics, $\wp$ is understood through degeneration of the σ -function.

The differential equation behind the Buchstaber genus

The logarithm determines the exponential without elliptic functions:

$$g'_{\text{Bc}}(u) = \frac{1}{\sqrt{1 - a_1 u^2 + a_2 u^4 - a_3 u^6}},$$

$$(f'_{\text{Bc}})^2 = 1 - a_1 f_{\text{Bc}}^2 + a_2 f_{\text{Bc}}^4 - a_3 f_{\text{Bc}}^6.$$

With $f(0) = 0$ and $f'(0) = 1$, differentiating gives

$$f'' = -a_1 f + 2a_2 f^3 - 3a_3 f^5.$$

This recursively determines every Taylor coefficient. For example,

$$f(u) = u - \frac{a_1}{6} u^3 + \frac{a_1^2 + 12a_2}{120} u^5 + O(u^7).$$

- ▶ The solution is odd, so $Q_{\text{Bc}}(u) = u/f_{\text{Bc}}(u)$ is even.
- ▶ Consequently the genus vanishes in odd complex dimensions and extends to oriented cobordism after inverting 2.

Theorem (Kornev, 2026): universality

- **Universality.** Over commutative $\mathbb{Z}[1/2]$ -algebras, F_{Bc} is universal among one-valued formal group laws whose formal inverse is $-u$ and whose modulus square is B_a .
- **Krichever–Ochanine specialization.** The law F_{Bc} belongs to Buchstaber's Krichever family $\mathcal{F}$ if and only if $a_3 = 0$. This specialization is the Ochanine genus.
- **Genus Bc.** The corresponding Hirzebruch genus is

$$Bc : \Omega_U \longrightarrow \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3\right], \quad \deg a_i = -4i.$$

The defining characteristic series first gives a rational genus; the integral formal group law gives the stated target.

[Kor26a]

Ochanine's elliptic quartic

Put $\delta = a_1/2$ and $\varepsilon = a_2$. Then

$$g_{\text{Och}}(u) = \int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}},$$

$$(f'_{\text{Och}})^2 = 1 - 2\delta f_{\text{Och}}^2 + \varepsilon f_{\text{Och}}^4.$$

- This is the invariant integral on the elliptic quartic $y^2 = 1 - 2\delta t^2 + \varepsilon t^4$, when the curve is nonsingular.
- The formal power series still make sense at singular parameters.

The intersection of the two families

Over $\mathbb{Q}$, the Buchstaber law F_{Bc} lies in the Buchstaber–Krichever family $\mathcal{F}$ precisely when $a_3 = 0$; this is the Ochanine genus.

Thus the third parameter records the departure from the Ochanine specialization. [Kor26a]

Classical genera as specializations of Bc

Genus	(a_1, a_2, a_3)	Exponential $f(u)$	$Q(u) = u/f(u)$
Signature L	$(2, 1, 0)$	$\tanh u$	$\frac{u}{\tanh u}$
$\hat{A}$	$(-\frac{1}{4}, 0, 0)$	$2 \sinh(u/2)$	$\frac{u}{2 \sinh(u/2)}$
Top Chern $\tilde{\chi}$	$(-3, 3, -1)$	$\frac{u}{\sqrt{1-u^2}}$	$\sqrt{1-u^2}$

- $\hat{A} = \text{Td}$ when $c_1 = 0$, and the two genera agree on St.
- $\tilde{\chi}|_{\text{St}} = \chi|_{\text{St}}$, where $\chi(M^{2d}) = \langle c_d(TM), [M] \rangle$ is the top Chern genus.

For an almost complex manifold, $\chi(M)$ is its Euler number.

The **Stong ring** $\text{St} \subset \Omega_{\mathbb{U}}$ is defined degreewise by

$$\text{St}^{-2d} = \{X \in \Omega_{\mathbb{U}}^{-2d} \mid c_l[X] = 0 \text{ for every } l \vdash d \text{ with an odd part}\}.$$

Here $c_l[X] = \langle c_{i_1} \cdots c_{i_s}, [X] \rangle$ is an ordinary tangent Chern number. Equivalently, $X \in \text{St}$ iff $2^r X \in \text{im}(\Omega_{\text{Sp}} \rightarrow \Omega_{\mathbb{U}})$ for some $r \geq 0$.

[HB]92, Buc78, BK26]

Theta divisors and $\mathbb{CP}^n$'s

Theta divisors

$$f_{\text{Bc}}(u) = u + \sum_{n \geq 1} \text{Bc}(\Theta_n) \frac{u^{n+1}}{(n+1)!}$$

$$\text{Bc}(\Theta_{2m+1}) = 0$$

$$\text{Bc}(\Theta_2) = -a_1$$

$$\text{Bc}(\Theta_4) = a_1^2 + 12a_2$$

$$\text{Bc}(\Theta_6) = -a_1^3 - 132a_1a_2 - 360a_3$$

Complex projective spaces

$$g_{\text{Bc}}(u) = \sum_{n \geq 0} \text{Bc}(\mathbb{CP}^n) \frac{u^{n+1}}{n+1}$$

$$\text{Bc}(\mathbb{CP}^{2m+1}) = 0$$

$$\text{Bc}(\mathbb{CP}^2) = \frac{a_1}{2}$$

$$\text{Bc}(\mathbb{CP}^4) = \frac{3a_1^2 - 4a_2}{8}$$

$$\text{Bc}(\mathbb{CP}^6) = \frac{5a_1^3 - 12a_1a_2 + 8a_3}{16}$$

Proposition (Kornev, 2026)

For every $n \geq 1$,

$$\text{Bc}(\Theta_n) \in \mathbb{Z}[a_1, a_2, a_3].$$

Thus no factor $1/2$ is needed for theta divisors.

[Kor26]

The Krichever genus and the Baker–Akhiezer function

The Krichever exponential is

$$f_{\text{Kr}}(u) = \frac{e^{\alpha u}}{\Phi(u, \rho)}, \quad \Phi(u, \rho) = \frac{\sigma(\rho - u)}{\sigma(\rho)\sigma(u)} e^{\zeta(\rho)u}.$$

Here $\zeta = \sigma'/\sigma$. Over $\mathbb{Q}$, the four parameters are

$$b_1 = \alpha, \quad b_2 = \wp(\rho), \quad b_3 = \wp'(\rho), \quad b_4 = G_2, \quad \deg b_i = -2i.$$

- Here u is the formal variable and ρ is the marked elliptic point. The relation $(\wp')^2 = 4\wp^3 - G_2\wp - G_3$ determines G_3 .
- In general $f_{\text{Kr}}(u) = u + \alpha u^2 + \cdots$ is not odd.

Different one-valued exponentials can give the same Buchstaber two-valued family under modulus square construction.

[Buc90, Kor26a]

Krichever and Buchstaber: the even part

Use $Y^2 = 4X^3 - G_2X - G_3$ and put $p = \wp(\rho)$, $r = \wp'(\rho)$. Then $r^2 = 4p^3 - G_2p - G_3$. Match parameters by

$$a_1 = -3p, \quad a_2 = 3p^2 - \frac{G_2}{4}, \quad a_3 = -\frac{r^2}{4}.$$

The normalized Buchstaber exponential becomes

$$f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u) - p}} = u + O(u^3).$$

Since $\Phi(u, \rho)\Phi(-u, \rho) = p - \wp(u)$, the Krichever exponential satisfies

$$f_{\text{Bc}}(u)^2 = -f_{\text{Kr}}(u)f_{\text{Kr}}(-u).$$

Therefore, for $Q(u) = u/f(u)$,

$$Q_{\text{Bc}}(u) = \sqrt{Q_{\text{Kr}}(u)Q_{\text{Kr}}(-u)}, \quad Q_{\text{Bc}}(0) = 1.$$

This removes every odd term of $\log Q_{\text{Kr}}$. In the earlier main-theorem notation, $G_2 = g_2$ and $G_3 = -g_3$. [Kor26a, DLMF]

A class in $\ker \text{Kr}$ detected by Bc

Proposition (Kornev, 2026)

There exists a class $K \in \Omega_{\text{U}}^{-12}$ such that

$$\text{Kr}(K) = 0 \quad \text{in } A \otimes \mathbb{Q}, \quad \text{Bc}(K) = -360a_3.$$

Real dimension 12 is minimal for $K \in \ker \text{Kr}$ with $\text{Bc}(K) \neq 0$.

This concerns separation of kernels. Matched genus values can already differ on SU -manifolds of real dimension 6.

Comparison with the Witten genus

The Witten genus is associated with the Weierstrass sigma-function

$$f_{\text{Wt}}(u) = \sigma(u; g_2, g_3).$$

For $a_1 = 0$, the Buchstaber exponential is

$$f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}.$$

The first nonzero coefficients are

$$f_{\text{Bc}}(u) = u - 3g_2 \frac{u^5}{5!} + 90g_3 \frac{u^7}{7!} + 189g_2^2 \frac{u^9}{9!} + O(u^{11}),$$

$$f_{\text{Wt}}(u) = u - \tilde{g}_2 \frac{u^5}{5!} - \tilde{g}_3 \frac{u^7}{7!} - 9\tilde{g}_2^2 \frac{u^9}{9!} + O(u^{11}),$$

where $\tilde{g}_2 = g_2/2$ and $\tilde{g}_3 = 6g_3$.

Hurwitz integrality

A series

$$f(u) = u + \sum_{n \geq 1} c_n \frac{u^{n+1}}{(n+1)!}, \quad c_n \in R,$$

is called a Hurwitz series over R ; we write $f(u) \in \mathcal{H}_R[[u]]$.

Proposition (Kornev, 2026)

For $a_1 = 0$,

$$f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}$$

is a Hurwitz series over $\mathbb{Z}[g_2, g_3]$.

The minimal Hurwitz ring of f_{Bc}

Theorem (Kornev, 2026)

For algebraically independent a_1, a_2, a_3 ,

$$f_{\text{Bc}} \in \mathcal{H}_{\mathbb{Z}[a_1, 12a_2, 360a_3]}[[u]],$$

and this is its minimal Hurwitz coefficient ring inside $\mathbb{Q}[a_1, a_2, a_3]$.

The key is the divided-power form of the differential equation:

$$f'' = -a_1 f + (12a_2) \frac{f^3}{3!} - (360a_3) \frac{f^5}{5!}.$$

- ▶ The coefficients of $f^k/k!$ are integral Bell polynomials in the Hurwitz coefficients of f , giving integrality by induction.
- ▶ The coefficients of $u^3/3!$, $u^5/5!$, and $u^7/7!$ recover a_1 , $12a_2$, and $360a_3$, proving minimality. In particular,

$$\mathbb{Z}[\text{Bc}(\Theta_n) \mid n \geq 1] = \mathbb{Z}[a_1, 12a_2, 360a_3].$$

Bunkova's minimal Hurwitz ring

Weierstrass obtained (1882) a recursive expansion of the elliptic sigma-function and proved that its normalized coefficients are integers. Equivalently,

$$\sigma(u; g_2, g_3) \in \mathcal{H}_{\mathbb{Z}[\frac{g_2}{2}, 2g_3]}[[u]].$$

Buchstaber–Leikin recovered this recursion from their heat-equation formalism and generalized the method to higher-genus sigma-functions. [BL05]

Bunkova asked whether the sharper coefficient ring $\mathbb{Z}[\frac{g_2}{2}, 6g_3]$ already suffices. [Bun17]

Proposition (Kornev, 2026)

$$\sigma(u; g_2, g_3) \in \mathcal{H}_{\mathbb{Z}[\frac{g_2}{2}, 6g_3]}[[u]].$$

The Witten exponential and its Hurwitz ring

For this comparison use the same Weierstrass invariants in both functions:

$$h(u) = \frac{1}{\sqrt{\wp(u; g_2, g_3)}}, \quad L(u) = \log \frac{\sigma(u; g_2, g_3)}{h(u)}, \quad A = \mathbb{Z}\left[\frac{g_2}{2}, 6g_3\right].$$

The Buchstaber result, with the sign of g_3 changed, gives $h \in \mathcal{H}_{\mathbb{Z}[3g_2, 90g_3]}[[u]] \subset \mathcal{H}_A[[u]]$.

Using $\zeta' = -\wp$ and the Weierstrass equation gives

$$L'' = \frac{g_2}{4}h^2 + \frac{g_3}{2}h^4 = \left(\frac{g_2}{2}\right) \frac{h^2}{2!} + 2(6g_3) \frac{h^4}{4!}, \quad L(0) = L'(0) = 0.$$

Integration and composition preserve Hurwitz series, hence

$$\sigma = he^L \in \mathcal{H}_A[[u]].$$

The coefficients of $u^5/5!$ and $u^7/7!$ are $-g_2/2$ and $-6g_3$; they also prove that A is minimal.

[Kor26a]

Two analytic variables and a formal variable

The coefficients used below are functions

$$\phi_{0,m}(\tau, \nu), \quad \tau \in \mathbb{H}, \quad \nu \in \mathbb{C}, \quad q = e^{2\pi i \tau}, \quad y = e^{2\pi i \nu}.$$

The subscript 0 is the modular weight; m is the Jacobi index.

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad P = [\nu] \in E_\tau.$$

- ▶ τ determines the elliptic curve, with periods normalized to 1, τ .
- ▶ ν represents a marked point. Jacobi forms are quasi-periodic in ν , so they represent sections of line bundles on E_τ .
- ▶ u in $Q(u) = u/f(u)$ is a separate **formal variable**; the coefficients of this series depend on (τ, ν) .

For an SU -manifold M^{2d} ,

$$\text{Ell}(M; \tau, \nu) \in J_{0,d/2}^{w,\mathbb{Z}}.$$

The condition $c_1(M) = 0$ supplies Jacobi modularity; it does not add a variable to a fixed genus.

[Gri99b, AG22]

Definitions of the four Jacobi generators

Use the three even theta functions and the normalized odd theta function:

$$\xi_{ab} = \frac{\theta_{ab}(\tau, \nu)}{\theta_{ab}(\tau, 0)}, \quad ab = 00, 01, 10, \quad \vartheta(\tau, 0) = 2\pi i \eta(\tau)^3.$$

All factors below have the same arguments (τ, ν) :

$$A = \phi_{0,1} = 4(\xi_{00}^2 + \xi_{01}^2 + \xi_{10}^2),$$

$$B = \phi_{0,2} = 2(\xi_{00}^2 \xi_{01}^2 + \xi_{00}^2 \xi_{10}^2 + \xi_{01}^2 \xi_{10}^2),$$

$$C = \phi_{0,3} = \left(\frac{\vartheta(\tau, 2\nu)}{\vartheta(\tau, \nu)} \right)^2, \quad D = \phi_{0,4} = \frac{\vartheta(\tau, 3\nu)}{\vartheta(\tau, \nu)}.$$

Their Fourier expansions start with

	A	B	C	D
$[q^0]$	$y + 10 + y^{-1}$	$y + 4 + y^{-1}$	$y + 2 + y^{-1}$	$y + 1 + y^{-1}$

In particular, $(A, B, C, D)(\tau, 0) = (12, 6, 4, 3)$. The displayed theta quotients extend holomorphically at lattice points.

[Gri99b, Gri99, AG22]

The elliptic genus on complex cobordism

Put $L = \mathbb{Z} + \tau\mathbb{Z}$ and

$$h = \frac{\vartheta(\tau, v)}{\vartheta_v(\tau, 0)}, \quad \ell(v) = \frac{\vartheta_v(\tau, v)}{\vartheta(\tau, v)}, \quad \gamma_1 = h\ell(v).$$

Gritsenko's theta expression has Chern-root factor

$$\tilde{Q}(x) = x \frac{\vartheta(\tau, v + x/(2\pi i))}{\vartheta(\tau, x/(2\pi i))}.$$

With $c = 2\pi i h$, the series $Q(u) = c^{-1} \tilde{Q}(cu)$ gives the same genus: the factors c^{-d} and c^d cancel in complex dimension d .

Normalized characteristic series and exponential

$$Q_{\text{Ell}}(u) = u \frac{\vartheta(\tau, v + hu)}{\vartheta(\tau, hu)}, \quad f_{\text{Ell}}(u) = \frac{\vartheta(\tau, hu)}{\vartheta(\tau, v + hu)}.$$

These define Ell on all of Ω_U . The condition $c_1 = 0$ is used for Jacobi modularity, not for this characteristic-series identity. [Gri99b]

The full Ell–Kr identity

Write ρ for the Krichever marked point. With $\eta_1 = \zeta_L(1/2)$,

$$\frac{\vartheta(\tau, z)}{\vartheta_\nu(\tau, 0)} = e^{-\eta_1 z^2} \sigma_L(z), \quad \ell(\nu) = \zeta_L(\nu) - 2\eta_1 \nu.$$

Recall the full Krichever exponential:

$$f_{\text{Kr}}(u; L, \rho, \alpha) = \frac{\sigma_L(u)\sigma_L(\rho)}{\sigma_L(\rho - u)} e^{(\alpha - \zeta_L(\rho))u}.$$

Substitute $\rho = -\nu$ and $\alpha = -\ell(\nu)$ in this expression.

Equality of normalized series

$$\begin{aligned} f_{\text{Ell}}(u) &= \frac{1}{h} f_{\text{Kr}}(hu; L, -\nu, -\ell(\nu)). \\ Q_{\text{Ell}}(u) &= Q_{\text{Kr}}(hu; L, -\nu, -\ell(\nu)). \end{aligned}$$

For every stably complex M^{2d} , this gives $\text{Ell}(M; \tau, \nu) = h^d \text{Kr}(M; L, -\nu, -\ell(\nu))$.

[Kri90, Bun22, DLMF]

The coefficient map is induced by the genus

Use $\mathcal{K} = \mathbb{Q}[\alpha, p, r, G_2]$, where $p = \wp_L(\rho)$, $r = \wp'_L(\rho)$. Define

$$H = \phi_{0,3/2} = \frac{\wp(\tau, 2v)}{\wp(\tau, v)} = -h^3 \wp'_L(v), \quad \gamma_1 = h\ell(v).$$

The exponential identity induces the homomorphism φ :

$$\begin{aligned} \varphi(\alpha) &= -\gamma_1, & \varphi(p) &= h^2 \wp_L(v) = \frac{A}{12}, \\ \varphi(r) &= H, & \varphi(G_2) &= h^4 G_2(L) = \frac{A^2}{12} - 2B. \end{aligned}$$

Thus $f_{\text{Ell}} = \varphi(f_{\text{Kr}})$ and $Q_{\text{Ell}} = \varphi(Q_{\text{Kr}})$, coefficient by coefficient.

$$\text{Ell} = \varphi \circ \text{Kr} \quad \text{on all of } \Omega_U.$$

For example, $\text{Kr}(\mathbb{CP}^1) = -2\alpha$ maps to $\text{Ell}(\mathbb{CP}^1) = 2\gamma_1$; the linear parameter is retained.

[Kri90, AG22, DLMF]

Ell and Kr genera in low degrees

For closed SU -manifolds, write $\langle P \rangle_M = \langle P, [M] \rangle$. The preceding map gives $\text{Ell}(M) = \varphi(\text{Kr}(M))$; $c_1 = 0$ removes α .

$$\text{Kr}(M^4) = p \langle c_2 \rangle_M, \quad \text{Ell}(M^4) = \frac{A}{12} \langle c_2 \rangle_M,$$

$$\text{Kr}(M^6) = \frac{r}{2} \langle c_3 \rangle_M, \quad \text{Ell}(M^6) = \frac{H}{2} \langle c_3 \rangle_M.$$

$$\text{Kr}(M^8) = p^2 \langle c_4 \rangle_M + \frac{G_2}{20} \langle c_2^2 - 2c_4 \rangle_M,$$

$$\text{Ell}(M^8) = \frac{A^2}{720} \langle 3c_2^2 - c_4 \rangle_M + \frac{B}{10} \langle 2c_4 - c_2^2 \rangle_M.$$

$$\begin{aligned} \text{Kr}(M^{12}) = & \frac{p^3}{7} \langle -c_2^3 + 3c_2c_4 - 2c_3^2 + 4c_6 \rangle_M \\ & + \frac{pG_2}{140} \langle 5c_2^3 - 8c_2c_4 + 3c_3^2 - 6c_6 \rangle_M \\ & + \frac{r^2}{28} \langle c_2^3 - 3c_2c_4 + 2c_3^2 + 3c_6 \rangle_M, \end{aligned}$$

$$\begin{aligned} \text{Ell}(M^{12}) = & \frac{A^3}{60480} \langle 10c_2^3 - 9c_2c_4 - c_3^2 + 2c_6 \rangle_M \\ & + \frac{AB}{840} \langle -5c_2^3 + 8c_2c_4 - 3c_3^2 + 6c_6 \rangle_M \\ & + \frac{H^2}{28} \langle c_2^3 - 3c_2c_4 + 2c_3^2 + 3c_6 \rangle_M. \end{aligned}$$

[Gri99, AG22, DLMF]

The full rational coefficient ring of Ell

The map φ is injective, as shown next. It identifies the rational images:

$$\mathcal{K} = \mathrm{Kr}(\Omega_U \otimes \mathbb{Q}) \xrightarrow[\varphi]{\sim} \mathcal{E}_{\mathbb{Q}} = \mathrm{Ell}(\Omega_U \otimes \mathbb{Q}) = \mathbb{Q}[\gamma_1, A, B, H].$$

Keeping all four integral-index Jacobi generators visible, the same ring is

$$\mathcal{E}_{\mathbb{Q}} \cong \frac{\mathbb{Q}[\gamma_1, A, B, C, D, H]}{(H^2 - C, 4D - AC + B^2)}.$$

In the cobordism grading,

	γ_1	A	H	B	C	D
deg	-2	-4	-6	-8	-12	-16

Here $C = H^2$ and $D = (AH^2 - B^2)/4$. On SU -manifolds the factor $e^{\gamma_1 c_1}$ disappears, leaving values in $\mathbb{Q}[A, B, H]$; odd complex dimensions can involve H .

The integral-index Jacobi ring $\mathbb{Q}[A, B, C]$ is a subring of the full target.

The integral image rings are isomorphic

Define the actual images of integral complex cobordism:

$$R_{\mathrm{Kr},\mathbb{Z}} = \mathrm{Kr}(\Omega_{\mathrm{U}}) \subset \mathcal{K}, \quad R_{\mathrm{Ell},\mathbb{Z}} = \mathrm{Ell}(\Omega_{\mathrm{U}}) \subset \mathcal{E}_{\mathbb{Q}}.$$

The identity $\mathrm{Ell}(M) = \varphi(\mathrm{Kr}(M))$ compares the value of the same bordism class on both sides. Since φ is injective,

Consequence for genus images

$$\varphi : R_{\mathrm{Kr},\mathbb{Z}} \xrightarrow{\sim} R_{\mathrm{Ell},\mathbb{Z}}, \quad \ker \mathrm{Ell} = \ker \mathrm{Kr}.$$

[Kri90, Bun22]

Associativity as a relation among four parameters

Put $e_1 = x + y + z$, $e_2 = xy + xz + yz$, and $e_3 = xyz$. Consider

$$P_2 = e_1^2 - 4e_2 + k_2e_3 + k_4e_1e_3 + k_6e_2e_3 + k_8e_3^2.$$

As a quadratic equation in z , $P_2 = 0$ defines a two-valued correspondence near the origin.

Theorem (Buchstaber–Kornev–Rubtsov–Veselov)

The associativity condition is

$$4k_8 = k_4^2 - k_2k_6.$$

Its integral parameter ring is

$$\mathcal{R} = \mathbb{Z}[k_2, k_4, k_6, k_8]/(4k_8 - k_4^2 + k_2k_6).$$

The symmetric formal-group interpretation is used after inverting 2. The integral presentation will connect this family to Jacobi forms.

[BV19, BKR26]

The same integral relation in Gritsenko's result

Let $A = \phi_{0,1}$, $B = \phi_{0,2}$, $C = \phi_{0,3}$, $D = \phi_{0,4}$. All four have modular weight 0; their Jacobi indices are 1, 2, 3, 4. Here $J_{0,*}^{w,\mathbb{Z}}$ denotes the ring with **integral indices**.

Theorem (Gritsenko, 1999)

$$J_{0,*}^{w,\mathbb{Z}} \cong \mathbb{Z}[A, B, C, D]/(4D - AC + B^2).$$

The generators A, B, C are algebraically independent.

The two-valued family has the following isomorphism of graded parameter rings:

$$\boxed{\mathcal{R} \xrightarrow{\sim} J_{0,*}^{w,\mathbb{Z}}}, \quad (k_2, k_4, k_6, k_8) \longmapsto (A, -B, C, -D),$$

where $\deg k_{2m} = m$ is the Jacobi index.

Over $\mathbb{Q}$, three generators suffice. Over $\mathbb{Z}$, the fourth generator records the divisibility of $AC - B^2$ by 4.

Jacobi forms specialize the Buchstaber parameters

The two presentations of the two-valued family are related by

$$k_2 = -4a_1, \quad k_4 = -2a_2, \quad k_6 = -4a_3, \quad k_8 = a_2^2 - 4a_1a_3.$$

Applying the same φ to $a_1 = -3p$, $a_2 = 3p^2 - G_2/4$, $a_3 = -r^2/4$ gives

$$\boxed{a_1 = -\frac{A}{4}, \quad a_2 = \frac{B}{2}, \quad a_3 = -\frac{C}{4}}.$$

For every closed oriented manifold M^{4m} , homogeneity gives

$$\mathrm{Bc}_J(M; \tau, \nu) = \mathrm{Bc}(M)|_{(a_1, a_2, a_3) = (-A/4, B/2, -C/4)} \in J_{0, m}^{w, \mathbb{Q}}.$$

Every monomial $a_1^{i_1} a_2^{i_2} a_3^{i_3}$ with $i_1 + 2i_2 + 3i_3 = m$ acquires Jacobi index m .

This is a Jacobi-valued oriented genus; no $c_1 = 0$ assumption is needed. Its rational Fourier coefficients need not be integral.

The Jacobi specialization of B_c is the even part

Use $h = \vartheta(\tau, v)/\vartheta_v(\tau, 0)$ and the normalized series

$$Q_{\text{Ell}}(u) = u \frac{\vartheta(\tau, v + hu)}{\vartheta(\tau, hu)}, \quad Q_{\text{Ell}}(0) = 1.$$

The sigma addition formula gives, with $p_v = \wp(v)$,

$$Q_{\text{Ell}}(u)Q_{\text{Ell}}(-u) = h^2 u^2 (\wp(hu) - p_v).$$

The Jacobi normalizations give

$$A = 12h^2 p_v, \quad B = 2h^4 \left(3p_v^2 - \frac{G_2}{4} \right), \quad C = h^6 \wp'(v)^2.$$

Thus the matched Buchstaber series satisfies

$$\boxed{Q_{B_{c_j}}(u) = \sqrt{Q_{\text{Ell}}(u)Q_{\text{Ell}}(-u)}}, \quad (a_1, a_2, a_3) = \left(-\frac{A}{4}, \frac{B}{2}, -\frac{C}{4} \right).$$

The square root has constant term 1; the theta normalization is $\vartheta_v(\tau, 0) = 2\pi i \eta(\tau)^3$.

[AG22, DLMF]

The complete odd logarithmic correction

Since $Q_{\text{Bc}_j}(u) = \sqrt{Q_{\text{Ell}}(u)Q_{\text{Ell}}(-u)}$, the ratio is determined by the odd logarithmic part:

$$\log \frac{Q_{\text{Ell}}(u)}{Q_{\text{Bc}_j}(u)} = \frac{1}{2} \log \frac{\vartheta(\tau, \nu + hu)}{\vartheta(\tau, \nu - hu)}.$$

For odd $k \geq 1$, define

$$\gamma_k(\tau, \nu) = h^k \left. \frac{\partial^k}{\partial w^k} \log \vartheta(\tau, w) \right|_{w=\nu}.$$

The derivative acts on ϑ ; $h = h(\tau, \nu)$ is held fixed. Taylor expansion gives the exact formal identity

$$Q_{\text{Ell}}(u) = Q_{\text{Bc}_j}(u) \exp \left(\sum_{r \geq 0} \gamma_{2r+1}(\tau, \nu) \frac{u^{2r+1}}{(2r+1)!} \right).$$

It retains every odd correction, beginning with the linear term.

The comparison in every dimension

Let M^{2d} be a closed stably complex manifold, with stable tangent Chern roots x_j . Define the characteristic classes

$$\mathcal{C}_{\text{BC}_j}(TM) = \prod_j Q_{\text{BC}_j}(x_j), \quad \text{ch}_k(TM) = \frac{1}{k!} \sum_j x_j^k.$$

Multiplying the series identity over all roots gives

General characteristic-class comparison

$$\text{Ell}(M; \tau, \nu) = \left\langle \mathcal{C}_{\text{BC}_j}(TM) \exp \left(\sum_{r \geq 0} \gamma_{2r+1}(\tau, \nu) \text{ch}_{2r+1}(TM) \right), [M] \right\rangle.$$

Only cohomological degree $2d$ contributes, so the sum is finite in each dimension. The factorials are already contained in ch_k .

- ▶ On SU -manifolds, $\text{ch}_1 = c_1 = 0$ removes $\gamma_1 c_1$.
- ▶ If all odd Chern-character classes vanish, the two genera agree.

No SU hypothesis is required for the displayed identity.

Agreement on symplectic cobordism and St

Quaternionic tangent Chern roots occur in pairs $x_i, -x_i$, so

$$Q_{\text{Ell}}(x_i)Q_{\text{Ell}}(-x_i) = Q_{\text{Bc}_J}(x_i)Q_{\text{Bc}_J}(-x_i).$$

Proposition (Kornev, 2026)

For every $X \in \text{St}$,

$$\text{Ell}(X) = \text{Bc}_J(X).$$

For general SU -manifolds, $c_1 = 0$ but higher odd Chern classes can survive.

[BK26]

The comparison in real dimensions 4, 8, and 10

For closed SU -manifolds, put $A = \phi_{0,1}$, $B = \phi_{0,2}$, and $H = \phi_{0,3/2}$.

$$\mathrm{Ell}(M^4) = \mathrm{Bc}_J(M^4) = \frac{A}{12} \langle c_2, [M] \rangle.$$

$$\begin{aligned} \mathrm{Ell}(M^8) = \mathrm{Bc}_J(M^8) &= \frac{A^2}{720} \langle 3c_2^2 - c_4, [M] \rangle \\ &\quad + \frac{B}{10} \langle 2c_4 - c_2^2, [M] \rangle. \end{aligned}$$

$$\mathrm{Bc}_J(M^{10}) = 0, \quad \mathrm{Ell}(M^{10}) = \frac{AH}{24} \langle c_5, [M] \rangle.$$

Here $\gamma_3 = H$ and $\gamma_5 = AH$. In real degree 10,

$$\frac{A}{12} c_2 \cdot \frac{H}{2} c_3 + \frac{AH}{24} (c_5 - c_2 c_3) = \frac{AH}{24} c_5.$$

The odd correction first contributes in a dimension divisible by 4 in real dimension 12. [AG22]

The correction in real dimension 12

Theta duplication gives

$$H = \frac{\vartheta(\tau, 2\nu)}{\vartheta(\tau, \nu)} = -h^3 \wp'(\nu), \quad H^2 = C.$$

Since $(\log \vartheta)''' = -\wp'$, the odd part is

$$\log Q_{\text{Ell}}(u) - \log Q_{\text{Bc}_J}(u) = \gamma_1 u + \frac{H}{6} u^3 + O(u^5).$$

Write $s_j = \sum_i x_i^j$ for tangent Chern roots. If $c_1 = 0$, then $s_1 = 0$ and $s_3 = 3c_3$. In complex degree six, the extra contribution is

$$\frac{1}{2} \left(\frac{H}{6} s_3 \right)^2 = \frac{C}{8} c_3^2.$$

Proposition (Kornev, 2026)

For every closed SU -manifold M^{12} ,

$$\text{Ell}(M) = \text{Bc}_J(M) + \frac{1}{8} \langle c_3^2, [M] \rangle \phi_{0,3}.$$

Information beyond the Ochanine genus and χ

Write $\text{Och} = \text{Bc}|_{a_3=0}$, and let χ be the top Chern number genus.

Theorem (Buchstaber–Kornev, 2026)

For every $n \geq 5$, there are $X_n, Y_n \in \Lambda^{-4n}$ such that

$$\text{Och}(X_n) = \text{Och}(Y_n) = 0, \quad \chi(X_n) = \chi(Y_n),$$

but

$$\text{Bc}(X_n) - \text{Bc}(Y_n) = 128(4a_1)^{n-5}a_3(a_1^2 - 3a_2) \neq 0.$$

The classes X_n , Y_n , and $X_n - Y_n$ are all indecomposable, even after rationalization.

Thus Bc detects information on St that the pair (Och, χ) does not determine.

[BK26]

Spin manifolds: the first obstruction in dimension 24

The genus extends to oriented cobordism by the composite, with $A = \mathbb{Z}[1/2, a_1, a_2, a_3]$,

$$\Omega_*^{SO} \longrightarrow \Omega_*^{SO}[1/2] \cong \Omega_*^{Sp}[1/2] \longrightarrow \Omega_*^U[1/2] \xrightarrow{\text{Bc}} A.$$

Call a value *integral* when it belongs to $\mathbb{Z}[a_1, a_2, a_3]$.

Theorem (Buchstaber–Kornev, 2026)

$\dim M < 24$: every spin (and hence Calabi–Yau) manifold M has integral $\text{Bc}(M)$;

$\dim M = 24$: every string class $[M]$ has integral $\text{Bc}(M)$,
but not every spin class does.

The Anderson–Brown–Peterson manifold. The closed spin manifold Σ_{ABP}^{24} has odd Pontryagin number $s_6(p_1, \dots, p_6)[\Sigma_{\text{ABP}}] \equiv 1 \pmod{2}$. Its existence follows from the ABP calculation of spin cobordism. It answered Milnor’s question and shows that his criterion first fails in dimension 24.

[Mil65, ABP67]

Cohomology theories for the Buchstaber genus

Put

$$\Delta = \text{Disc}_t(t^3 - a_1 t^2 + a_2 t - a_3).$$

Theorem (Kornev, 2026)

1. There exists a complex-oriented multiplicative commutative cohomology theory

$$\text{Bc}^*(-)[\Delta^{-1}], \quad \text{Bc}^*(\text{pt})[\Delta^{-1}] \cong \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3, \Delta^{-1}\right],$$

whose orientation induces the localized Buchstaber genus.

2. There exists a complex-oriented multiplicative commutative cohomology theory $\text{Bc}^*(-)$ whose coefficient homomorphism realizes

$$\text{Bc} : \Omega_{\text{U}} \longrightarrow \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3\right].$$

Both theories are obtained by the method of Landweber, Ravenel, Stong, and Mironov.

[LRS95]

Conclusions: the four genera

1. The Buchstaber exponential satisfies $(f'_{\text{Bc}})^2 = 1 - a_1 f_{\text{Bc}}^2 + a_2 f_{\text{Bc}}^4 - a_3 f_{\text{Bc}}^6$. Its oddness links the genus to quaternionic and oriented cobordism.
2. The specialization $a_3 = 0$ is Ochanine's genus. The degree-12 class K separates Bc from Krichever's genus by $\text{Bc}(K) = -360a_3$.
3. The minimal Hurwitz rings are $\mathbb{Z}[a_1, 12a_2, 360a_3]$ for f_{Bc} and $\mathbb{Z}[g_2/2, 6g_3]$ for the Witten exponential σ .
4. Spin and string integrality lead to topological applications; complex-oriented multiplicative cohomology theories realize Bc.
5. Bc is the even multiplicative symmetrization of Kr. The faithful map φ gives $\text{Ell} = \varphi \circ \text{Kr}$ and identifies their integral genus images.

Thank you for your attention!

Selected references 1 of 3

- [Kor26a] M. Kornev, *Buchstaber, Ochanine, Krichever, and Witten Genera*, arXiv:2603.21118 (expanded version).
- [BK26] V. Buchstaber and M. Kornev, *Logarithm of the Universal Two-Valued Formal Group*, arXiv:2609.15878 (2026).
- [Kor26] M. I. Kornev, *Buchstaber, Ochanine, Krichever, and Witten genera*, J. Geom. Phys. 226 (2026), 105855.
- [BV24] V. Buchstaber and A. Veselov, *Chern–Dold character and theta divisors*, Adv. Math. 449 (2024), 109720.
- [Kor, in prep.] M. Kornev, *n-Valued Analogue of Abel’s Theorem*, preprint in preparation.
- [Abel26] N. H. Abel, *Untersuchung der Functionen zweier unabhängig veränderlichen Größen x und y , wie $f(x, y)$, welche die Eigenschaft haben, daß $f(z, f(x, y))$ eine symmetrische Function von z, x und y ist*, J. Reine Angew. Math. 1 (1826), 11–15.
- [Haz78] M. Hazewinkel, *Formal Groups and Applications*, Pure Appl. Math. 78, Academic Press, 1978.
- [BN71] V. M. Buchstaber and S. P. Novikov, *Formal groups, power systems and Adams operators*, Math. USSR-Sb. 13 (1971), 80–116.
- [Buc75] V. M. Buchstaber, *Two-valued formal groups. I*, Math. USSR-Izv. 9 (1975), 987–1006.

Selected references 2 of 3

- [Buc78] V. M. Buchstaber, *Topological applications of two-valued formal groups*, Math. USSR-Izv. 12 (1978), 125–177.
- [Buc90] V. M. Buchstaber, *Functional equations and two-valued algebraic groups*, Russian Math. Surveys 45:3 (1990), 213–215.
- [BL05] V. M. Buchstaber and D. V. Leikin, *Addition laws on Jacobian varieties*, Proc. Steklov Inst. Math. 251 (2005), 49–120.
- [Bun17] E. Y. Bunkova, *Weierstrass sigma-function coefficients divisibility hypothesis*, arXiv:1701.00848.
- [Bun22] E. Yu. Bunkova, *Isomorphism of coefficient rings of Buchstaber and Krichever formal groups*, arXiv:2212.13483 (2022).
- [Mil65] J. Milnor, *On the Stiefel–Whitney numbers of complex manifolds and of spin manifolds*, Topology 3 (1965), 223–230.
- [ABP67] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, *The structure of the spin cobordism ring*, Ann. of Math. 86 (1967), 271–298.
- [LRS95] P. S. Landweber, D. C. Ravenel, and R. E. Stong, *Periodic cohomology theories defined by elliptic curves*, Contemp. Math. 181 (1995), 317–337.
- [Wit88] E. Witten, *The index of the Dirac operator in loop space*, Lecture Notes in Math. 1326 (1988).
- [HBJ92] F. Hirzebruch, T. Berger, and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics E20, Vieweg, 1992.

Selected references 3 of 3

- [BV19] V. M. Buchstaber and A. P. Veselov, *Conway topograph, $\mathrm{PGL}_2(\mathbb{Z})$ -dynamics and two-valued groups*, Russian Math. Surveys 74:3 (2019), 387–430.
- [BKR26] V. Buchstaber, M. Kornev, and V. Rubtsov, *Two-Valued Groups, Chazy Equation, Dubrovin–Frobenius Structures, and QYBE*, arXiv:2604.26641 (2026).
- [Gri99] V. Gritsenko, *Complex vector bundles and Jacobi forms*, arXiv:math/9906191 (1999).
- [GW20] V. Gritsenko and H. Wang, *Graded rings of integral Jacobi forms*, arXiv:1810.09392, revised 2020.
- [BL00] L. A. Borisov and A. Libgober, *Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex*, arXiv:math/0007126 (2000).
- [AG22] D. Adler and V. Gritsenko, *Elliptic genus and modular differential equations*, J. Geom. Phys. 181 (2022), 104662; arXiv:2209.00038.
- [Gri99b] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms*, arXiv:math/9906190 (1999).
- [DLMF] NIST Digital Library of Mathematical Functions, *Weierstrass elliptic and modular functions*, Chapter 23.
- [Kri90] I. M. Krichever, *Generalized elliptic genera and Baker–Akhiezer functions*, Math. Notes 47 (1990), 132–142.