

Buchstaber Genus, Formal Groups, and Cohomology Theories

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One-dimensional n -valued formal groups

Buchstaber and Novikov introduced two-valued formal groups in connection with complex and symplectic cobordism, as well as the **modulus-square construction** from one-valued to two-valued formal groups. [BN71]

Definition

Let R be a commutative ring. A one-dimensional n -valued formal group is encoded by a monic polynomial

$$P(Z; x, y) = Z^n - \theta_0(x, y)Z^{n-1} + \cdots + (-1)^n \theta_{n-1}(x, y) \in R[[x, y]][[Z]].$$

Its roots define the unordered product

$$x * y = [z_1, \dots, z_n], \quad P(z_j; x, y) = 0,$$

which is associative, has a unit 0 with $x * 0 = 0 * x = [x, \dots, x]$, and admits an inverse $\text{inv}(x)$ with $0 \in x * \text{inv}(x)$ and $0 \in \text{inv}(x) * x$.

Buchstaber's two-valued classification

Theorem (Buchstaber, 1975)

Let R be an integral-domain $\mathbb{Q}$ -algebra. Every commutative two-valued formal group is strongly isomorphic either to the elementary cyclic law

$$\mathbb{G}_2 : \quad x * y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2],$$

or to the diagonal law

$$x * y = [x + y, x + y].$$

If z_1, z_2 are the roots of $P(Z; x, y)$, a strict series $B(x) = x + O(x^2)$ gives a strong isomorphism $P \rightsquigarrow Q$ when

$$\prod_{j=1}^2 (Z - B(z_j)) = Q(Z; B(x), B(y)).$$

For $Q = \mathbb{G}_2$, the series B is called the **logarithm**; Buchstaber found it explicitly.

Commutativity and cyclic classification

Earlier results

- ▶ Kholodov classified three-valued formal groups up to strong isomorphism.
- ▶ For every $n \leq 11$, he obtained only finitely many elementary laws, i.e. laws with homogeneous $P(Z; x, y)$.
- ▶ In the classical work, commutativity was assumed by default.

The commutativity statement is specific to the formal setting: A. Levine constructed a noncommutative two-valued group on six points.

Theorem (Kornev, 2026)

Every n -valued formal group over a reduced $\mathbb{Q}$ -algebra is commutative.

Theorem (Kornev, 2026): classification in generic case

Let R be an integral-domain $\mathbb{Q}$ -algebra and suppose

$$\text{Disc}_Z P(Z; x, y) \neq 0 \quad \text{in } R[[x, y]].$$

Then P is strictly isomorphic over R to the cyclic law

$$\mathbb{G}_n : \quad x * y = \left[\left(\sqrt[n]{x} + \zeta_n^k \sqrt[n]{y} \right)^n \right]_{k=0}^{n-1}.$$

From associativity to an additive coordinate

Wenn man z. B. die Functionen $x + y$ und xy durch $f(x, y)$ bezeichnet, so ist für die erste $f(z, f(x, y)) = z + f(x, y) = z + x + y$ und für die zweite $f(z, f(x, y)) = z \cdot f(x, y) = zxy$. Die Function $f(x, y)$ hat also in den beiden Fällen die merkwürdige Eigenschaft, daß $f(z, f(x, y))$ eine symmetrische Function der drei unabhängig-veränderlichen Größen z, x und y ist. Ich will in diesem Aufsätze die allgemeine Gestalt der Functionen suchen, welche eben diese Eigenschaft haben.

Figure: Abel's opening paragraph [Abel26].

Abel assumed the stronger condition that $F(z, F(x, y))$ is symmetric in x, y, z .

For a one-dimensional formal group law, we start only from **associativity** and a two-sided unit.

Differentiate associativity at the unit



obtain an invariant differential



integrate it to get an additive coordinate

$$g(F(x, y)) = g(x) + g(y)$$

Hence, in characteristic zero, commutativity follows from associativity and the unit.

[Haz78]

Weak units and polynomial pullbacks

Example

The multiplication

$$x * y = [x + y, x - y]$$

defines an associative two-valued formal group. Since $x * 0 = [x, x]$ and $0 * x = [x, -x]$, the element 0 is a weak unit, but not a strong unit.

weak unit: $x \in 0 * x$ and $x \in x * 0$

strong unit: $0 * x = x * 0 = [x, \dots, x]$

Proposition (Kornev, 2026)

Let $P(Z; x, y) = 0$ define an n -valued formal group with strong unit 0, and let $T \in \mathbb{C}[z]$ be monic of degree $m \geq 2$ with $T(0) = 0$. Then

$$P_T(Z; x, y) := P(T(Z); T(x), T(y)) = 0$$

defines an associative mn -valued formal group with weak unit 0. If P is commutative, then so is P_T .

Example: the Fermat curve. Pulling back the additive formal group by $T(z) = z^n$ gives

$$P_{\mathbb{F}_n}(Z; x, y) = Z^n - x^n - y^n, \quad x *_{\mathbb{F}_n} y = \left[\zeta_n^k \sqrt[n]{x^n + y^n} \right]_{k=0}^{n-1}.$$

Thus $0 * y = y * 0 = [\zeta_n^k y]_{k=0}^{n-1}$, so 0 is weak but not strong for $n > 1$.

Buchstaber's universal two-valued family

In 1990 Buchstaber introduced

$$\mathcal{B}_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz), \quad \mathbf{a} = (a_1, a_2, a_3).$$

Each $\mathcal{B}_a$ defines a two-valued formal group and is universal among modulus squares of the formal groups

$$\mathcal{F}(u, v) = \frac{u^2\lambda_1(v) - v^2\lambda_1(u)}{u\lambda_2(v) - v\lambda_2(u)}.$$

[Buc90]

Buchstaber–Krichever family $\mathcal{F} \xrightarrow{\text{modulus square}} \mathcal{B}_a \xleftarrow{\text{modulus square}} \text{formal group } F_{\mathcal{B}_a}$

The modulus-square construction is not injective: distinct one-valued formal groups can determine the same two-valued law.

Main Theorem: the formal group F_{Bc}

Put

$$R_a(t) = 1 - a_1 t^2 + a_2 t^4 - a_3 t^6.$$

► Then

$$F_{Bc}(u, v) = \left[\left(\frac{u\sqrt{R_a(v)} - v\sqrt{R_a(u)}}{u^2 - v^2} \right)^2 + a_3 u^2 v^2 \right]^{-1/2}$$

is a one-dimensional formal group law with formal inverse $\bar{u} = -u$.

- Its modulus square is exactly the two-valued law $\mathcal{B}_a(z; x, y)$.
- Its exponential is odd, a fact used below for the values on theta divisors.

Main Theorem: explicit logarithms and exponentials

► Define

$$g_2 = 4 \left(\frac{a_1^2}{3} - a_2 \right), \quad g_3 = 4 \left(\frac{a_1 a_2}{3} - \frac{2a_1^3}{27} - a_3 \right).$$

Two-valued law $\mathcal{B}_a$

$$B(x) = \left(\int_0^{\sqrt{x}} \frac{dt}{\sqrt{1 + a_1 t^2 + a_2 t^4 + a_3 t^6}} \right)^2 \quad B^{-1}(x) = \frac{1}{\wp(\sqrt{x}; g_2, g_3) - a_1/3}.$$

One-valued law F_{Bc}

$$g_{Bc}(u) = \int_0^u \frac{dt}{\sqrt{1 - a_1 t^2 + a_2 t^4 - a_3 t^6}} \quad f_{Bc}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3) + a_1/3}}.$$

For singular Weierstrass cubics, $\wp$ is understood through degeneration of the σ -function.

Main Theorem: universality

- **Universality.** Over integral-domain $\mathbb{Z}[1/2]$ -algebras, F_{Bc} is universal among one-valued formal group laws whose formal inverse is $-u$ and whose modulus square is $\mathcal{B}_a$.
- **Krichever–Ochanine specialization.** The law F_{Bc} belongs to Buchstaber's Krichever family $\mathcal{F}$ if and only if $a_3 = 0$. This specialization is the Ochanine genus.
- **Genus Bc.** The corresponding Hirzebruch genus is

$$\text{Bc} : \Omega_{\text{U}} \longrightarrow \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3\right], \quad \deg a_i = -4i.$$

Recall that by default any genus lands on $\mathbb{Q}[a_1, a_2, a_3]$.

Theta divisors and $\mathbb{CP}^n$'s

Theta divisors

$$f_{\text{Bc}}(u) = u + \sum_{n \geq 1} \text{Bc}(\Theta_n) \frac{u^{n+1}}{(n+1)!}$$

$$\text{Bc}(\Theta_{2m+1}) = 0$$

$$\text{Bc}(\Theta_2) = -a_1$$

$$\text{Bc}(\Theta_4) = a_1^2 + 12a_2$$

$$\text{Bc}(\Theta_6) = -a_1^3 - 132a_1a_2 - 360a_3$$

Complex projective spaces

$$g_{\text{Bc}}(u) = \sum_{n \geq 0} \text{Bc}(\mathbb{CP}^n) \frac{u^{n+1}}{n+1}$$

$$\text{Bc}(\mathbb{CP}^{2m+1}) = 0$$

$$\text{Bc}(\mathbb{CP}^2) = \frac{a_1}{2}$$

$$\text{Bc}(\mathbb{CP}^4) = \frac{3a_1^2 - 4a_2}{8}$$

$$\text{Bc}(\mathbb{CP}^6) = \frac{5a_1^3 - 12a_1a_2 + 8a_3}{16}$$

Proposition (Kornev, 2026)

For every $n \geq 1$,

$$\text{Bc}(\Theta_n) \in \mathbb{Z}[a_1, a_2, a_3].$$

Thus no factor $1/2$ is needed for theta divisors.

[Kor26]

Comparison with the Krichever genus in degree 12

Proposition (Kornev, 2026)

There exists a class $K \in \Omega_U^{-12}$ such that

$$\mathrm{Kr}(K) = 0 \quad \text{in } A \otimes \mathbb{Q}, \quad \mathrm{Bc}(K) = -360a_3.$$

Moreover, degree 12 is the first degree in which such a class can occur.

Thus the two genera are already distinguished by a complex-cobordism class in degree 12.

Comparison with the Witten genus

The Witten genus is associated with the Weierstrass sigma-function

$$f_{\text{Wt}}(u) = \sigma(u; g_2, g_3).$$

For $a_1 = 0$, the Buchstaber exponential is

$$f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}.$$

The first nonzero coefficients are

$$f_{\text{Bc}}(u) = u - 3g_2 \frac{u^5}{5!} + 90g_3 \frac{u^7}{7!} + 189g_2^2 \frac{u^9}{9!} + O(u^{11}),$$

$$f_{\text{Wt}}(u) = u - \tilde{g}_2 \frac{u^5}{5!} - \tilde{g}_3 \frac{u^7}{7!} - 9\tilde{g}_2^2 \frac{u^9}{9!} + O(u^{11}),$$

where $\tilde{g}_2 = g_2/2$ and $\tilde{g}_3 = 6g_3$.

Hurwitz integrality

A series

$$f(u) = u + \sum_{n \geq 1} c_n \frac{u^{n+1}}{(n+1)!}, \quad c_n \in R,$$

is called a Hurwitz series over R ; we write $f(u) \in HR[[u]]$.

Proposition (Kornev, 2026)

For $a_1 = 0$,

$$f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}$$

is a Hurwitz series over $\mathbb{Z}[g_2, g_3]$.

Bunkova's minimal Hurwitz ring

Weierstrass obtained (1882) a recursive expansion of the elliptic sigma-function and proved that its normalized coefficients are integers. Equivalently,

$$\sigma(u; g_2, g_3) \in H\mathbb{Z}\left[\frac{g_2}{2}, 2g_3\right][[u]].$$

Buchstaber–Leikin recovered this recursion from their heat-equation formalism and generalized the method to higher-genus sigma-functions. [BL05]

Bunkova asked whether the sharper coefficient ring $\mathbb{Z}\left[\frac{g_2}{2}, 6g_3\right]$ already suffices. [Bun17]

Proposition (Kornev, 2026)

$$\sigma(u; g_2, g_3) \in H\mathbb{Z}\left[\frac{g_2}{2}, 6g_3\right][[u]].$$

Rings associated with the universal two-valued law

In complex cobordism, the universal modulus-square law has the form

$$Z^2 - \Theta_1(x, y)Z + \Theta_2(x, y) = 0.$$

- Λ the subring of Ω_U generated by the coefficients of Θ_1 and Θ_2 ;
- $\tilde{\Lambda}$ the subring of $\Lambda[1/2]$ generated by the coefficients of $\frac{1}{2}\Theta_1$ and Θ_2 ;
- $\mathcal{L}$ the coefficient ring of Buchstaber's functor.

$$\mathcal{L}^*(X) = \text{Hom}_{A^U}(U^*(M\text{Sp}), U^*(X)), \quad \mathcal{L} = \mathcal{L}^*(\text{pt}) \subset \Omega_U.$$

Buchstaber's description of $\mathcal{L}$

Theorem (Buchstaber, 1978)

Inside $\Omega_U \otimes \mathbb{Q}$,

$$\mathcal{L} = \tilde{\Lambda} \cap \Omega_U, \quad \tilde{\Lambda} = \mathbb{Z}[c_n \mid n \geq 1], \quad \deg c_n = -4n.$$

The Adams–Novikov spectral sequence for $M\mathrm{Sp}$ implies

$$\mathcal{L} = \Omega_{\mathrm{Sp}} \left[\frac{1}{2} \right] \cap \Omega_U.$$

Thus $\mathcal{L}$ is the **saturation** (servant extension) of $\Omega_{\mathrm{Sp}}/\mathrm{Tors} \subset \Omega_U$: if a nonzero integer multiple of $x \in \Omega_U$ comes from symplectic cobordism, then $x \in \mathcal{L}$.

The Stong ring

Let $\text{St} \subset \Omega_U$ be Stong's subring, characterized by the vanishing of the characteristic numbers indexed by partitions containing an odd part.

Theorem (Buchstaber, 1978)

$$\mathcal{L} = \text{St}$$

Properties

- ▶ $\mathcal{L}$ is the saturation of $\Omega_{\text{Sp}}/\text{Tors}$ inside Ω_U .
- ▶ $\mathcal{L} = \tilde{\Lambda} \cap \Omega_U$.
- ▶ All prescribed odd-part characteristic numbers vanish.

Rationally

$$\begin{aligned}\mathcal{L} \otimes \mathbb{Q} &\cong \Omega_{\text{Sp}} \otimes \mathbb{Q} && \text{as graded rings} \\ \mathcal{L} \otimes \mathbb{Q} &\cong \Omega_U^{\text{even}} \otimes \mathbb{Q} && \text{as graded } \mathbb{Q}\text{-vector spaces}\end{aligned}$$

Geometric generators and examples

$$\mathcal{L} \otimes \mathbb{Q} \cong \mathbb{Q}[\xi_1, \xi_2, \dots], \quad \xi_n = [Y^{[n]}].$$

Here Y is any smooth projective surface with $c_1^2(Y) = 0$ and $c_2(Y) \neq 0$; for example, a K3 surface.

- ▶ smooth projective holomorphic symplectic varieties;
- ▶ Hilbert schemes $S^{[n]}$ and generalized Kummer varieties $K_n(A)$;
- ▶ smooth projective Calabi–Yau fourfolds and products of these varieties.

Integral values on the Stong ring

Theorem (Buchstaber–Kornev, 2026).

The restriction of the Buchstaber genus to $\mathcal{L} = \text{St}$ takes values in

$$\text{Bc}(\mathcal{L}) \subset \mathbb{Z}[2a_1, a_2, 2a_3].$$

In particular, the restriction to $\mathcal{L}$ takes values in $\mathbb{Z}[a_1, a_2, a_3]$, without inverting 2. Although

$$\text{Bc}(\Theta_n) \in \mathbb{Z}[a_1, a_2, a_3],$$

none of the positive even-dimensional theta-divisor classes belongs to $\mathcal{L}$:

$$[\Theta_{2m}] \notin \mathcal{L}, \quad m \geq 1.$$

Indeed, $c_1^{2m}[\Theta_{2m}] = (2m + 1)! \neq 0$.

Euler and Todd genera via oddification

The Euler characteristic and the Todd genus do not arise directly from odd exponentials. For a Hirzebruch genus with characteristic series $Q(x)$, define its **oddification** by

$$\tilde{Q}(x) = \sqrt{Q(x)Q(-x)}, \quad \tilde{Q}(0) = 1.$$

Theorem (Buchstaber–Kornev, 2026)

The genera determined by Q and $\tilde{Q}$ have the same restriction to the Stong ring:

$$\varphi_{\tilde{Q}}|_{\text{St}} = \varphi_Q|_{\text{St}}.$$

Examples

$$\begin{aligned} Q_\chi(x) &= 1 + x, & \tilde{Q}_\chi(x) &= \sqrt{1 - x^2}, \\ Q_{\text{Td}}(x) &= \frac{x}{1 - e^{-x}}, & \tilde{Q}_{\text{Td}}(x) &= \frac{x}{2 \sinh(x/2)}. \end{aligned}$$

Corollary (Buchstaber–Kornev, 2026)

	(a_1, a_2, a_3)	$Q_{\text{Bc}} \text{ spec'n.}$
χ	$(-3, 3, -1)$	$\sqrt{1 - x^2}$
Td	$(-\frac{1}{4}, 0, 0)$	$\frac{x}{2 \sinh(x/2)}$

These specializations of Bc coincide on St with χ and Td, respectively.

Spin manifolds: the first obstruction in dimension 24

The genus extends to oriented cobordism by the composite, with $A = \mathbb{Z}[1/2, a_1, a_2, a_3]$,

$$\Omega_*^{\mathrm{SO}} \longrightarrow \Omega_*^{\mathrm{SO}}[1/2] \cong \Omega_*^{\mathrm{Sp}}[1/2] \longrightarrow \Omega_*^{\mathrm{U}}[1/2] \xrightarrow{\mathrm{Bc}} A.$$

Call a value *integral* when it belongs to $\mathbb{Z}[a_1, a_2, a_3]$.

Theorem (Kornev, 2026)

$\dim M < 24$: every spin (and hence Calabi–Yau) manifold M has integral $\mathrm{Bc}(M)$;

$\dim M = 24$: every string class $[M]$ has integral $\mathrm{Bc}(M)$,
but not every spin class does.

The Anderson–Brown–Peterson manifold. The closed spin manifold $\Sigma_{\mathrm{ABP}}^{24}$ has odd Pontryagin number $s_6(p_1, \dots, p_6)[\Sigma_{\mathrm{ABP}}] \equiv 1 \pmod{2}$. Its existence follows from the ABP calculation of spin cobordism. It answered Milnor’s question and shows that his criterion first fails in dimension 24.

[Mil65, ABP67]

Cohomology theories for the Buchstaber genus

Put

$$\Delta = \text{Disc}_t(t^3 - a_1 t^2 + a_2 t - a_3).$$

Theorem (Kornev, 2026)

1. There exists a complex-oriented multiplicative commutative cohomology theory

$$\text{Bc}^*(-)[\Delta^{-1}], \quad \text{Bc}^*(\text{pt})[\Delta^{-1}] \cong \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3, \Delta^{-1}\right],$$

whose orientation induces the localized Buchstaber genus.

2. There exists a complex-oriented multiplicative commutative cohomology theory $\text{Bc}^*(-)$ whose coefficient homomorphism realizes

$$\text{Bc} : \Omega_{\text{U}} \longrightarrow \mathbb{Z}\left[\frac{1}{2}, a_1, a_2, a_3\right].$$

Both theories are obtained by the method of Landweber, Ravenel, Stong, and Mironov.

[LRS95]

Conclusions

1. Reduced one-dimensional n -valued formal groups over $\mathbb{Q}$ are commutative. In the generically separable case they are strictly isomorphic to the cyclic law.
2. The main theorem gives F_{Bc} , its logarithm and exponential, and its universality. The specialization $a_3 = 0$ is the Ochanine case.
3. The values on theta divisors and on $\mathbb{CP}^n$ are explicit. Moreover, $\text{Bc}(\Theta_n) \in \mathbb{Z}[a_1, a_2, a_3]$, and for $a_1 = 0$ the exponential is Hurwitz over $\mathbb{Z}[g_2, g_3]$.
4. The Stong ring satisfies $\mathcal{L} = \text{St}$ and $\text{Bc}(\mathcal{L}) \subset \mathbb{Z}[2a_1, a_2, 2a_3]$, although $[\Theta_{2m}] \notin \mathcal{L}$ for every $m \geq 1$.
5. The oriented extension is integral on every spin manifold of dimension < 24 . In dimension 24, every string manifold M has integral $\text{Bc}(M)$, but this is false for arbitrary spin manifolds.
6. The genus is realized by complex-oriented multiplicative commutative cohomology theories, before and after inverting Δ .

Selected references

- [Abel26] N. H. Abel,
Untersuchung der Functionen zweier unabhängig veränderlichen Größen x und y , wie $f(x, y)$, welche die Eigenschaft haben, daß $f(z, f(x, y))$ eine symmetrische Function von z, x und y ist, J. Reine Angew. Math. 1 (1826), 11–15.
- [Haz78] M. Hazewinkel,
Formal Groups and Applications, Pure Appl. Math. 78, Academic Press, 1978.
- [Mil65] J. Milnor,
On the Stiefel–Whitney numbers of complex manifolds and of spin manifolds, Topology 3 (1965), 223–230.
- [BN71] V. M. Buchstaber and S. P. Novikov,
Formal groups, power systems and Adams operators, Math. USSR-Sb. 13 (1971), 80–116.
- [Buc75] V. M. Buchstaber,
Two-valued formal groups. I, Math. USSR-Izv. 9 (1975), 987–1006.
- [Buc78] V. M. Buchstaber,
Topological applications of two-valued formal groups, Math. USSR-Izv. 12 (1978), 125–177.
- [Buc90] V. M. Buchstaber,
Functional equations and two-valued algebraic groups, Russian Math. Surveys 45:3 (1990), 213–215.
- [ABP67] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson,
The structure of the spin cobordism ring, Ann. of Math. 86 (1967), 271–298.
- [BV24] V. Buchstaber and A. Veselov,
Chern–Dold character and theta divisors, Adv. Math. 449 (2024), 109720.
- [BL05] V. M. Buchstaber and D. V. Leikin,
Addition laws on Jacobian varieties, Proc. Steklov Inst. Math. 251 (2005), 49–120.
- [Bun17] E. Y. Bunkova,
Weierstrass sigma-function coefficients divisibility hypothesis, arXiv:1701.00848.
- [LRS95] P. S. Landweber, D. C. Ravenel, and R. E. Stong,
Periodic cohomology theories defined by elliptic curves, Contemp. Math. 181 (1995), 317–337.
- [Wit88] E. Witten,
The index of the Dirac operator in loop space, Lecture Notes in Math. 1326 (1988).
- [Kor26] M. I. Kornev,
Buchstaber, Ochanine, Krichever, and Witten genera, J. Geom. Phys. 226 (2026), 105855.



Thank you for your attention!