

Fermat Hypersurfaces, Projective Duality, and Algebraic n -Valued Monoids*

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Fermat's Hypersurfaces

For each $d, n \geq 1$ define a *Fermat hypersurface*:

$$M_d^n = \{F_d^n = 0\} = \{z_0^d + \dots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1} =: \mathbb{P}^{n+1}.$$

The complete linear system $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ gives the Veronese embedding of degree d (with $N = \binom{n+d+1}{d} - 1$ below):

$$\nu_d : \mathbb{P}^{n+1} \hookrightarrow \mathbb{P}((H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))^*) \cong \mathbb{P}^N$$

$$(z_0 : \dots : z_{n+1}) \mapsto (Y_{d, \dots, 0} : \dots : Y_{0, \dots, d}), \quad Y_{d_0, \dots, d_{n+1}} = z_0^{d_0} \dots z_{n+1}^{d_{n+1}}, \\ d_0 + d_1 + \dots + d_{n+1} = d.$$

We have

$$M_d^n = \nu_d(\mathbb{P}^{n+1}) \cap H_{F_d^n},$$

where $H_{F_d^n}$ denotes the hyperplane

$$H_{F_d^n} = \left\{ [Y] \in \mathbb{P}^N \mid \sum_{j=0}^{n+1} Y_{0, \dots, 0, \underbrace{d}_{j}, 0, \dots, 0} = 0 \right\}.$$

First Examples of Varieties M_d^n

We have the 2-parameter family of *smooth, irreducible* projective varieties:

$$M_d^n = \{z_0^d + \dots + z_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}$$

$$M_1^n \cong \mathbb{P}^n, \quad n \geq 1$$

$$M_d^1 \text{ is a curve of genus } (d-1)(d-2)/2, \quad d \geq 1$$

$$M_2^n \cong \mathrm{SO}(n+2)/(\mathrm{SO}(2) \times \mathrm{SO}(n)), \quad n \geq 1$$

$$M_2^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$$

$$M_2^4 \cong \mathrm{Gr}_{\mathbb{C}}(2, 4)$$

Euler Characteristic of M_d^n

Using the exact sequence for the normal bundle $\mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}}$

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^{n+1}}|_X \longrightarrow \mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}} \longrightarrow 0$$

and the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus n+2} \longrightarrow \mathcal{T}_{\mathbb{P}^{n+1}} \longrightarrow 0$$

(with $\mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}} \cong \mathcal{O}_X(d)$), we have:

Corollary [Dim92, Ch. 5, §3, Exercise 3.7(i)]

For any smooth complex hypersurface X of degree d in $\mathbb{P}^{n+1}$,

$$\chi(X) = n + 2 + \frac{(1 - d)^{n+2} - 1}{d}.$$

H^k , H_k , and π_k for Hyperplane Sections

Recall the following classical result:

Lefschetz Hyperplane Theorem [Voi03, Theorem 1.23]

Let $X \subset \mathbb{P}^N$ be an n -dimensional complex projective algebraic variety, let Y be a hyperplane section of X such that $U = X \setminus Y$ is smooth, and let $i : Y \hookrightarrow X$ be the embedding.

Then the induced maps are iso's for $k < n - 1$ and for $k = n - 1$:

$$H_k(Y; \mathbb{Z}) \xrightarrow{i_*} H_k(X; \mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^k(X; \mathbb{Z}) \xrightarrow{i^*} H^k(Y; \mathbb{Z})$$

$$\pi_k(Y, \mathbb{Z}) \xrightarrow{i_*} \pi_k(X, \mathbb{Z}) \longrightarrow 0$$

(Co-)Homology of M_d^n

Corollary

$$H_k(M_d^n; \mathbb{Z}) \cong H^k(M_d^n; \mathbb{Z}) \cong$$

$$\cong \begin{cases} \mathbb{Z}, & k \text{ even, } k \neq n, \text{ and } 0 \leq k \leq 2n, \\ 0, & k \text{ odd,} \\ \mathbb{Z}^{b_n(d)}, & k = n. \end{cases}$$

$$b_n(d) = \begin{cases} \chi(M_d^n) - n = 2 + \frac{(1-d)^{n+2} - 1}{d}, & n \text{ even,} \\ n + 1 - \chi(M_d^n) = \frac{1 - (1-d)^{n+2}}{d} - 1, & n \text{ odd.} \end{cases}$$

Cohomology Ring of M_d^n

Let $i : M_d^n \hookrightarrow \mathbb{P}^{n+1}$, $H^2(M_d^n, \mathbb{Z}) = \mathbb{Z}h$, where $h = i^*c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$, $u = [M_d^n] \in H^{2n}(M_d^n, \mathbb{Z})$, and Q be the intersection form. Define

$$H^n(M_d^n)_{\text{van}} = \text{Ker}(L : H^n(M_d^n, \mathbb{Z}) \xrightarrow{\cup h} H^{n+2}(M_d^n, \mathbb{Z})) .$$

For n odd, $H^{n+2}(M_d^n) = 0$, so $H^n(M_d^n) \cong H^n(M_d^n)_{\text{van}}$.

For n even, $H^n(M_d^n) \cong \mathbb{Z}h^{n/2} \oplus H^n(M_d^n)_{\text{van}}$.

$$\left\{ \begin{array}{ll} h^k \cup h^\ell = h^{k+\ell}, & \text{for each } 0 \leq k, \ell, k + \ell \leq n. \\ h^n = du. \\ h^k = 0, & \text{for each } k > n. \\ h \cup \alpha = 0, & \text{for each } \alpha \in H^n(M_d^n)_{\text{van}}. \\ \alpha \cup \beta = Q(\alpha, \beta)u, & \text{for each } \alpha, \beta \in H^n(M_d^n)_{\text{van}}. \end{array} \right.$$

Homotopy Groups of M_d^n

Corollary

Let $d \geq 2$, $n \geq 3$. The computations of $\pi_k(M_d^n)$ for $k \leq n - 1$ are as follows:

$$\pi_k(M_d^n) \cong \begin{cases} 0, & k = 1, n \geq 3, \\ \mathbb{Z}, & k = 2, n \geq 3, \\ 0, & 3 \leq k \leq n - 1, n \geq 4. \end{cases}$$

M_d^1 is a curve of genus $(d - 1)(d - 2)/2$. Also, $M_1^n = \mathbb{C}P^{n-1}$. The cases of the surfaces M_d^2 and the threefolds M_d^3 will be considered soon.

E_8 form

Consider the surface

$$M_4^2 = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{P}^3.$$

Its intersection form $Q_{M_4^2}$ on $H^2(M_4^2; \mathbb{Z})$ is even, unimodular, of signature -16 ,

$$Q_{M_4^2} \cong 3H \oplus 2(-E_8),$$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_8 = \begin{pmatrix} 2 & & & & & & & \\ & 1 & & & & & & \\ & & 2 & & & & & \\ & & & 1 & & & & \\ & & & & 2 & & & \\ & & & & & 1 & & \\ & & & & & & 2 & \\ & & & & & & & 1 \end{pmatrix}$$

In particular, $\dim_+ Q_{M_4^2} = 3$, $\dim_- Q_{M_4^2} = 19$, and $b_2(M_4^2) = 22$.

M_4^2 is a K3 Surface

A K3 surface is a compact complex surface that is simply connected and has trivial first Chern class (Calabi–Yau surface).

There is a *unique diffeomorphism type* of K3 surface, although the moduli space of all complex K3 surfaces has complex dimension 20 [Sco05].

$c_1 = 0$	1
$b_0 = b_4 = 1$	0 0
$b_1 = b_3 = 0$	1 20 1
$\chi = 24$	0 0
$\tau = -16$	1
$b_2 = \chi - 2 = 22$	

The Hodge diamond
of any K3 surface

K3 Surfaces



The real part of a K3 surface. Reproduced from [Dol20]

Surfaces M_d^2

From the Freedman's classification of 4-manifolds, we get:

Proposition

If d is odd, the variety M_d^2 is or.-preserving homeomorphic to

$$(\mathbb{C}P^2)^{\#m} \# (\overline{\mathbb{C}P^2})^{\#n}, \quad m = \frac{1}{3}(d^3 - 6d^2 + 11d - 3),$$

$$n = \frac{1}{3}(d - 1)(2d^2 - 4d + 3).$$

If d is even, M_d^2 is spin and it is or.-preserving homeomorphic to

$$(\mathbb{K}3)^{\#m} \# (\mathbb{C}P^1 \times \mathbb{C}P^1)^{\#n}, \quad m = \frac{d(d^2 - 4)}{48},$$

$$n = \frac{1}{48}(d - 4)(13d^2 - 44d + 12).$$

The Problem of

$$\Omega_{SO}^{-4} = \langle [\mathbb{C}P^2] \rangle_{\mathbb{Z}}$$

$$\Omega_U^{-4} = \langle [\mathbb{C}P^1]^2, [\mathbb{C}P^2] \rangle_{\mathbb{Z}}$$

From the [Buc70, Theorem 4.11], we obtain:

Proposition (Buchstaber, Kornev, 2025)

For each $d \geq 1$, we have in Ω_U :

$$[M_d^2] = \frac{d(d-1)^2}{2} [\mathbb{C}P^1]^2 + \frac{d(4-d^2)}{3} [\mathbb{C}P^2].$$

We see that $[M_d^2]_{\Omega_U} \mapsto \frac{d(4-d^2)}{3} [\mathbb{C}P^2]_{\Omega_{SO}}$ under the natural homomorphism

$$\Omega_U^{-4} \longrightarrow \Omega_{SO}^{-4}.$$

The Problem of

So, in Ω_U , we have the identification:

$$[M_d^2] = \begin{cases} (m-n)[\mathbb{C}P^2] + \frac{d(d-1)^2[\mathbb{C}P^1]^2}{2}, & d \text{ is odd,} \\ m[\mathbb{K}3] + n[\mathbb{C}P^1]^2 - \frac{(d-4)(d-2)(7d-6)[\mathbb{C}P^1]^2}{48}, & d \text{ is even} \end{cases}$$

The terms in **blue** correspond to the "naïve" case of complex structures glued along connected sums: $[M_1 \# M_2] = [M_1] + [M_2]$.

The terms in **magenta** describe the discrepancy provided by the complex structure of M_d^2 .

$[M_d^2]$ Via Θ -Divisors

Using [BV24, formula (6)], we get

Proposition (Buchstaber, Kornev, 2025)

$$[M_d^2] = d \cdot \left(\frac{(d^2 - 4)}{6} [\Theta_2] + \frac{(5 - 2d)}{2} [\Theta_1]^2 \right),$$

where Θ_j is a smooth theta divisor of a general principally polarised abelian variety A^{j+1} , considered as the complex manifold of real dimension $2j$.

For $d = 4$: $[M_4^2] = 2 \cdot (4[\Theta_2] - 3[\Theta_1]^2)$.

For $d = 5$: $2[M_5^2] = 5 \cdot (7[\Theta_2] - 5[\Theta_1]^2)$.

For $d = 7$: $2[M_7^2] = 21 \cdot (5[\Theta_2] - 3[\Theta_1]^2)$.

Classification of 6-Manifolds

Let M be a closed oriented simply connected 6-manifold with

$$H^3(M, \mathbb{Z}) = 0 \text{ and } \text{Tors } H^*(M, \mathbb{Z}) = 0.$$

We have the following data:

- a free abelian group $H^2(M)$,
- a symmetric trilinear map $\mu : \text{Sym}^3(H^2(M)) \longrightarrow \mathbb{Z}$,
 $\mu(x, y, z) = \langle x \cup y \cup z, [M] \rangle$,
- a linear form $p : H^2(M) \longrightarrow \mathbb{Z}$, $p(x) = \langle p_1(M) \cup x, [M] \rangle$,
where $p_1(M)$ is the first Pontryagin class,
- the second Stiefel–Whitney class $w_2(M) \in H^2(M, \mathbb{Z}/2)$.

Classification of 6-Manifolds

Theorem [Wal66, Jup73, Zhu80]

There exists an orientation-preserving diffeomorphism $f : M' \rightarrow M$ if and only if there is an isomorphism

$$\varphi : H^2(M, \mathbb{Z}) \longrightarrow H^2(M', \mathbb{Z})$$

such that

$$\mu'(\varphi(x), \varphi(y), \varphi(z)) = \mu(x, y, z), \quad p'(\varphi(x)) = p(x), \quad \overline{\varphi}(w_2) = w'_2.$$

In the case of quasitoric manifolds, under certain assumptions on the moment polytopes (see [BEM⁺17]), f is a diffeomorphism iff there exists a graded ring isomorphism

$$\psi : H^*(M, \mathbb{Z}) \rightarrow H^*(M', \mathbb{Z}).$$

The Milnor–Hirzebruch Problem

The Milnor–Hirzebruch problem was first posed by Hirzebruch in his ICM 1958 talk [Hir58]. Its algebraic version can be formulated as follows:

Which sets of $p(n)$ characteristic numbers

$$c_\lambda, \lambda \in \mathcal{P}(n)$$

can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n ?

This version of the problem remains largely open, although some arithmetic restrictions are known from the theorem of Grothendieck–Riemann–Roch–Hirzebruch.

The Manifold N_d^3

Theorem [Wal66]

Let M be a closed smooth 1-connected 6-manifold.
Then we can write M as a connected sum (up to diffeomorphism)

$$M \cong M_1 \# M_2,$$

where $H_3(M_1)$ is finite and M_2 is diffeomorphic to a connected sum of copies of $\mathbb{S}^3 \times \mathbb{S}^3$.

From this result, we get the orientation preserving diffeomorphism

$$M_d^3 \cong N_d^3 \# (\mathbb{S}^3 \times \mathbb{S}^3)^{b_3(M_d^3)/2}.$$

We have $H^*(N_d^3) \cong H^*(\mathbb{CP}^3)$ additively, $p_1(N_d^3) = (5 - d^2)x^2$, $p_1(\mathbb{CP}^3) = 4h^2$, where $H^2(N_d) = \mathbb{Z}x$, $H^2(\mathbb{CP}^3) = \mathbb{Z}h$.

So, N_d^3 is $\mathbb{CP}^3$ -like.

The Manifold N_d^3

Corollary

For any $d \geq 3$ there exists a smooth 6-dimensional homology projective space N_d^3 (i.e. $H^*(N_d^3) \cong H^*(\mathbb{C}P^3)$ additively) such that

$$p_1(N_d^3) = (5 - d^2)x^2,$$

where x is the generator of $H^2(N_d^3, \mathbb{Z})$.

The Manifold N_d^3

Proposition (Buchstaber, Kornev, 2025)

The manifold N_d^3 is Kähler if and only if $d \in \{1, 2\}$.

$$M_1^3 = N_1^3 \cong \mathbb{C}P^3.$$

$M_2^3 = N_2^3 \cong \mathrm{SO}(5)/(\mathrm{SO}(2) \times \mathrm{SO}(3))$ is an algebraic variety.

There exists an additive isomorphism

$$H^*(N_d^3, \mathbb{Z}) \cong H^*(\mathbb{C}P^3, \mathbb{Z})$$

which is *not* a ring isomorphism for $d > 1$.

Proof. Assume, for contradiction, that N_d^3 is Kähler when $d \geq 3$. Write

$$c_1 = rx, \quad c_2 = \alpha x^2$$

for $r, \alpha \in \mathbb{Z}$. From one hand,

$$p_1(N_d^3) = (5 - d^2)x^2.$$

From the other,

$$p_1(N_d^3) = c_1^2 - 2c_2.$$

So, we get

$$r^2 - 2\alpha = 5 - d^2. \tag{1}$$

From the Hirzbruch–Riemann–Roch theorem

$$\chi(\mathcal{O}_{N_d^3}) = \langle \text{td}(\mathcal{T}_{N_d^3}), [N_d^3] \rangle,$$

we know that $\langle c_1 c_2 / 24, [N_d^3] \rangle = \chi(\mathcal{O}(N_d^3)) = \sum_{q=0}^3 (-1)^q h^{0,q} = 1$.

Hence, we have another equation:

$$r\alpha d = 24. \quad (2)$$

From (1) and (2), one has $\alpha > 0$ and $r > 0$ when $d \geq 3$. So, $-K_{N_d^3} = rx$ is ample since $c_1 = rx$, $x \in H^{1,1}$ and $r > 0$. Thus, N_d^3 is projective (by Kodaira and Chow theorems) and Fano. The Picard number:

$$\rho = \text{rk NS}(N_d^3) = \text{rk } H^2(X; \mathbb{Q}) = 1.$$

Solving the system of equations (1) and (2), we get the following set of solutions

$$(r, d) \in \{(4, 1), (3, 2), (2, 3), (1, 4)\}.$$

From the classification of Fano threefolds [PI99, Ch. 12, §12.2], one get $b_3(N_d^3) \neq 0$ giving the contradiction:

$$-K_{N_d^3}^3 = \langle c_1(N_d^3)^3, N_d^3 \rangle = r^3 d$$

For $(r, d) = (2, 3)$ case, we have $-K_{N_d^3}^3 = 24$. But $h^{1,2} \neq 0$.

For $(1, 4)$ case, we have $-K_{N_d^3}^3 = 4$. But again $h^{1,2} \neq 0$.



r	$-K_X^3$	g	$h^{1,2}$	X	R	U
1	2	2	52	$X \rightarrow \mathbb{P}^3$ is a double cover with the ramification divisor of degree 6	-	?
1	4	3	30	$X = X_4 \subset \mathbb{P}^4$ is a quartic, or $X \rightarrow Q \subset \mathbb{P}^5$ is a double cover of a non-singular quadric with the ramification divisor of degree 8	-	*
1	6	4	20	$X = X_6 \subset \mathbb{P}^5$ is a complete intersection of a quadric and a cubic	-	+
1	8	5	14	$X = X_8 \subset \mathbb{P}^6$ is a complete intersection of three quadrics	-	+
1	10	6	10	$X = X_{10} \subset \mathbb{P}^7$ is a section of $\text{Gr}(2, 5)$ embedded by Plücker by a subspace of codimension 2 and a quadric, or a section of a cone $W = W_5 \subset \mathbb{P}^7$ over a non-singular del Pezzo threefold $V = V_5 \subset \mathbb{P}^6$ of degree 5 by a quadric	**	+
1	12	7	7	$X = X_{12} \subset \mathbb{P}^8$	+	+
1	14	8	5	$X = X_{14} \subset \mathbb{P}^9$ is a section of the Grassmannian $\text{Gr}(2, 6)$ embedded by Plücker by a subspace of codimension 5	-	+
1	16	9	3	$X = X_{16} \subset \mathbb{P}^{10}$	+	+
1	18	10	2	$X = X_{18} \subset \mathbb{P}^{11}$	+	+
1	22	12	0	$X = X_{22} \subset \mathbb{P}^{13}$	+	+
2	8		21		-	?
2	8 · 2		10	a finite cover of $\mathbb{P}^3$ of degree 2 ramified along a smooth surface $B \subset \mathbb{P}^3$ of degree 4	-	+
2	8 · 3		5	$X = X_3 \subset \mathbb{P}^4$, a smooth cubic	-	+
2	8 · 4		2	$X = X_4 \subset \mathbb{P}^5$, a smooth intersection of two quadrics	+	+
2	8 · 5		0	$X = X_5 \subset \mathbb{P}^6$, a section of the Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a subspace of codimension 3	+	+
3	27 · 2		0	$Q \subset \mathbb{P}^4$, a smooth quadric	+	+
4	64		0	$\mathbb{P}^3$	+	+

A connected compact Kähler manifold M of complex dimension n is said to be a *fake projective space* if M has the same Betti numbers as $\mathbb{C}P^n$ but is different from $\mathbb{C}P^n$.

First examples of fake projective space were obtained in complex dimension 2 by Mumford [Mum79]. A classification of fake projective planes was given in [PY07].

Theorem [Yeu10]

- 1 There is no fake projective three space M of general type.
- 2 There is no fake projective three space M with the same rational cohomology ring as $\mathbb{C}P^3$.
- 3 Any fake projective threefold has to be Fano.
- 4 A complete list of Fano fake projective threefolds is known and can be found in [IŠ83]. Examples are given by hyperquadrics in $\mathbb{C}P^4$.

The Manifold N_d^3

One can induce the stable almost complex structure from $M_d^3 \# (\mathbb{S}^3 \times \mathbb{S}^3)^{\#m}$ to the summands so that $s_3(\mathbb{S}^3 \# \mathbb{S}^3) = 0$. Indeed, it is possible because $\pi_5(BU) = 0$ and hence there is no obstruction to extend a map $\mathbb{S}^5 \rightarrow BU$ to a map $\mathbb{D}^6 \rightarrow BU$. Moreover, these almost complex structures are in bijection with $\pi_6(BU) \cong \mathbb{Z}$.

Proposition (Buchstaber, Kornev, 2025)

For each $d \geq 1$, there exists smooth stably complex 6-manifold N_d such that:

- N_d^3 is a homology $\mathbb{C}P^3$.
- $[M_d^3] = [N_d^3]$ in Ω_U .
- N_d^3 cannot be Kähler for $d \geq 3$.

The Milnor–Hirzebruch Problem and M_d^3

Once again, from [Buc70, Theorem 4.11], we obtain:

Proposition

Let $A = [\mathbb{C}P^1]$, $B = [\mathbb{C}P^2]$, and $C = [\mathbb{C}P^3]$ in Ω_U^{-6} . Then we get

$$24[M_d^3] = -d \left(15d(d-1)^2 A^3 - 20(d-1)^2 (d+1) AB - 6(5-d^3) C \right).$$

For $d = 1$: $[M_1^3] = [\mathbb{C}P^3]$.

For $d = 2$: $2[M_2^3] = -5A^3 + 10AB - 3C$.

For $d = 5$: $[M_5^3] = -50(5A^3 - 8AB + 3C)$.

Recall that $\Omega_{SO}^{-6} = 0$.

$[M_d^3]$ Via Θ -Divisors

Proposition (Buchstaber, Kornev, 2025)

$$[M_d^3] = d \cdot \left(\frac{(d^3 - 5)}{24} [\Theta_3] - \frac{5(d + 3)(d - 2)}{12} [\Theta_2 \times \Theta_1] + \frac{5(3d - 7)}{8} [\Theta_1]^3 \right),$$

where Θ_j is a smooth theta divisor of a general principally polarised abelian variety A^{j+1} , considered as the complex manifold of real dimension $2j$.

For $d = 5$: $[M_5^3] = 25 \cdot ([\Theta_3] - 2[\Theta_2 \times \Theta_1] + [\Theta_1]^3)$.

For $d = 7$: $12[M_7^3] = 7 \cdot (169[\Theta_3] - 250[\Theta_2 \times \Theta_1] + 105[\Theta_1]^3)$.

For $d = 11$: $4[M_{11}^3] = 11 \cdot (221[\Theta_3] - 210[\Theta_2 \times \Theta_1] + 65[\Theta_1]^3)$.

The Sullivan Conjecture

Manifolds M_d^n are examples of complete intersections.

Let $i : X_n(\mathbf{d}) \subset \mathbb{C}P^{n+k}$ be the transverse intersection of k complex hypersurfaces of degree $\mathbf{d} = \{d_1, \dots, d_k\}$.

By an observation of Thom, the diffeomorphism type of $X_n(\mathbf{d})$ depends only on the multidegree $\mathbf{d}$.

The Sullivan data associated to the complete intersection $X_n(\mathbf{d})$ is the tuple

$$SD_n(\mathbf{d}) := \left(d, (p_i(n, \mathbf{d}))_{i=1}^{\lfloor n/2 \rfloor}, \chi(X_n(\mathbf{d})) \right) \in \mathbb{Z}^+ \times \mathbb{Z}^{\lfloor n/2 \rfloor} \times \mathbb{Z},$$

where $d = d_1 \dots d_k$, $p_i(X(\mathbf{d})) = p_i(n, \mathbf{d})x^{2i}$ — Pontryagin numbers, $\chi(-)$ is Euler's characteristic, and $x = i^*c_1(\mathbb{C}P^{n+k})$.

The Sullivan Conjecture

The Sullivan Conjecture. Suppose that $n \geq 3$ and $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are complete intersections. If $SD_n(\mathbf{d}) = SD_n(\mathbf{d}')$, then $X_n(\mathbf{d})$ is diffeomorphic to $X_n(\mathbf{d}')$.

For $n = 1$, it holds. But the converse conjecture not.

For $n = 2$, it is too difficult: the problem with smooth structures. The converse conjecture is also false: $X_2(4)$, $X_2(3, 2)$ and $X_2(2, 2, 2)$ are all K3-surfaces.

For $n \geq 3$, the converse conjecture holds [CN25, Proposition 2.10].

For $n = 3$, it follows from [Wal66] and [Jup73].

For $n = 4$: yes, [CN25, Theorem 1.3].

The Sullivan Conjecture

If $5 \leq n \leq 7$, then Fang and Wang (2010) proved that the Sullivan conjecture holds up to homeomorphism.

The complete intersections

$$X_4(3^{(150)}, 7^{(89)}, 9^{(65)}, 15, 25^{(130)}) \quad \text{and} \quad X_4(5^{(261)}, 21^{(89)}, 27^{(64)})$$

(where $3^{(150)}$ stands for 150 copies of 3, etc.) are diffeomorphic [CN25, Example 1.5].

Basics of Projective Duality

Consider a projective space $\mathbb{P} = \mathbb{P}(V_{\mathbb{C}})$. Hyperplanes in $\mathbb{P}$ form the dual projective space $\mathbb{P}^* = \mathbb{P}(V^*)$.

Let $X \subset \mathbb{P}$ be a closed *irreducible* algebraic subvariety.

A hyperplane $H \subset \mathbb{P}$ is said to be tangent to X if there exists a smooth point $x \in X$ such that $x \in H$ and the tangent space to H at x contains the tangent space to X at x , i.e. $\mathcal{T}_x X \subset \mathcal{T}_x H$.

Denote by $X^\vee \subset \mathbb{P}^*$ the closure of the set of all hyperplanes tangent to X . The variety $X^\vee$ is called *projectively dual* to X .

Basics of Projective Duality



The Coffee Cup Caustic.

Grace Weir, 2005

Reproduced from graceweir.com/page31/page31.html

Basics of Projective Duality

- Let $H = \{\ell = 0\}$. Then

x is a singular point of $X \cap H \Leftrightarrow d\ell|_{\mathcal{T}_x X} = 0 \Leftrightarrow \mathcal{T}_x X \subset \mathcal{T}_x H$.

If X is smooth and does not lie in any hyperplane, then

$H \in X^\vee \Leftrightarrow \exists x \in X$ with $\mathcal{T}_x X \subset \mathcal{T}_x H \Leftrightarrow H \cap X$ is singular.

- For any projective variety $X \subset \mathbb{P}$ we have $(X^\vee)^\vee = X$.
- If $z \in X$ and $H \in X^\vee$ are smooth points, then

H is tangent to X at $z \Leftrightarrow z \subset \mathbb{P}^*$ is tangent to $X^\vee$ at H .

- If X is irreducible then $X^\vee$ is irreducible.

Basics of Projective Duality

For any smooth curve $X \subset \mathbb{P}^2$ of degree d , according to Plücker formulas [GKZ94, Proposition 2.4], we have $\deg X^\vee = d(d-1)$.

Hence, $X^\vee$ cannot be smooth for $\deg X > 3$ because $(X^\vee)^\vee = X$.

Basics of Projective Duality

Suppose that X is a hypersurface (not a hyperplane) in $\mathbb{P}(V)$. Then $X^\vee = \{\Delta_X = 0\}$ is a hypersurface in $\mathbb{P}(V^*)$. We shall call the *X -discriminant* the defining polynomial Δ_X of $X^\vee$.

Example. For $M_2^1 = \{x^2 + y^2 = z^2\}$, we have

$$(M_2^1)^\vee = \{u^2 + v^2 + w^2 = 0\}.$$

Example [GKZ94]. For the curve $M_3^1 = \{x^3 + y^3 = z^3\}$ we have

$$(M_3^1)^\vee = \{u^6 + v^6 + w^6 - 2u^3v^3 - 2u^3w^3 - 2v^3w^3 = 0\}.$$

The curve $(M_3^1)^\vee$ is not smooth since M_3^1 is smooth and $\deg M_3^1 = 3 > 2$.

n -Valued Groups: a Brief History

In 1971, V. M. Buchstaber and S. P. Novikov proposed a construction *motivated by the theory of characteristic classes* [BN71].

This construction describes a multiplication in which *the product of any pair of elements is a multiset of n points* [Buc06].

An axiomatic definition of n -valued groups and the first results of their algebraic theory were obtained in a subsequent series of works by V. M. Buchstaber.

Currently, the theory of n -valued (formal, finite, discrete, topological, and algebro-geometric) groups and their applications in various areas of mathematics and mathematical physics are being developed by a number of authors.

n -Valued Monoids

An *n -valued monoid* is a space X equipped with an operation

$$* : X \times X \rightarrow \text{Sym}^n(X)$$

where $\text{Sym}^n(X) = X^{\times n}/S_n$ is the space of unordered n -tuples of elements of X :

- *Associativity.* The n^2 -multisets

$$\begin{aligned} &[x * w \mid w \in y * z], \\ &[w * z \mid w \in x * y] \end{aligned}$$

coincide.

- *Unit.* There exists an element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for every $x \in X$.

n -Valued Groups

An *n -valued group* X is an n -valued monoid equipped with an inverse map

$$\text{inv} : X \rightarrow X$$

that is, a map such that for every $x \in X$

$$x * \text{inv}(x) \ni e, \quad \text{inv}(x) * x \ni e.$$

The notions of homomorphisms, commutativity, kernels, and related concepts admit natural generalizations from the context of 1-valued groups to that of n -valued groups.

Algebraic n -Valued Monoids and Groups

An *algebraic n -valued monoid* is an algebraic variety X equipped with an associative n -valued multiplication given by a *rational morphism* $X \times X \rightarrow \text{Sym}^n(X)$ with a neutral element $e \in X$ such that

$$x * e = e * x = [x, x, \dots, x] \quad \text{for every } x \in X.$$

An *algebraic n -valued group* is an algebraic n -valued monoid on X together with a *regular morphism* $\text{inv} : X \rightarrow X$ such that for any $x \in X$ the following two conditions hold:

$$e \in x * \text{inv}(x), \quad x * \text{inv}(x) = \text{inv}(x) * x.$$

Buchstaber Polynomials

Consider a 3-parametrized family of polynomials:

$$B_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz)$$

In elementary symmetric functions:

$$B_a(z; x, y) = (e_1^2 - 4e_2) - 4a_1e_3 - 2a_2e_1e_3 - 4a_3e_2e_3 + (a_2^2 - 4a_1a_3)e_3^2.$$

The polynomials $B_a(z; x, y)$ endow $\mathbb{C}$ with the structure of the universal regular algebraic 2-valued group $\mathbb{G}_{\mathbb{C}}(B_a)$ for any $\mathbf{a} \in \mathbb{C}^3$, with addition

$$x * y = [z \mid B_a(z; x, y) = 0],$$

neutral element 0, and inversion map $\text{inv}(x) = x$ [BV19, Theorem 6.3], [BK25b, Theorem 1].

Weierstrass Form of a Cubic

Let $\delta_a = \Delta_t(t^3 + a_1t^2 + a_2t + a_3)$ be the discriminant. Then

$$\delta_a = -4a_3a_1^3 + a_2^2a_1^2 + 18a_2a_3a_1 - 4a_2^3 - 27a_3^2.$$

Let $\mathcal{E}$ be an irreducible cubic over the field $\mathbb{C}$. As is well known (see, e.g., [FW69, Exercise 5–24]), $\mathcal{E}$ is isomorphic to a cubic given in $\mathbb{C}P^2$ with coordinates $(x : y : z)$, in the chart $\{z = 1\}$, by

$$\mathcal{E} = \{y^2 = x^3 + a_1x^2 + a_2x + a_3\}.$$

As an abelian variety over $\mathbb{C}$, a cubic admits only automorphisms of orders 2, 3, 4, and 6 [Har77, Corollary 4.7].

2-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Gaiur, Rubtsov) [BGR24]

(I) If $\delta_a \neq 0$, the algebraic 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ with identity ∞ is the regular coset group $\mathcal{E}_{\langle \sigma \rangle}$ for the group of points of the elliptic curve $\mathcal{E}$ with respect to the involution $\sigma : (x, y) \mapsto (x, -y)$. The group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ on $\mathbb{CP}^1$ has zero $(0 : 1)$ and addition

$$\mu_a : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^2$$

$$(x_1 : x_0) * (y_1 : y_0) = (u_2 : u_1 : u_0)$$

$$\begin{cases} u_2 = (x_1 y_0 - x_0 y_1)^2, \\ -\frac{1}{2} u_1 = x_1 x_0 (2a_1 y_1 y_0 + a_2 y_1^2 + y_0^2) + x_1^2 y_1 (a_2 y_0 + 2a_3 y_1) + x_0^2 y_0 y_1, \\ u_0 = x_1^2 y_1 (a_2^2 y_1 - 4a_3(a_1 y_1 + y_0)) - 2x_0 x_1 y_1 (a_2 y_0 + 2a_3 y_1) + x_0^2 y_0^2. \end{cases}$$

Theorem (Buchstaber, Kornev, 2025)

(II) The isomorphism class of the coset algebraic 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ is completely determined by the j -invariant of the elliptic curve $\mathcal{E}$:

$$j(a) = 6912 \frac{(3a_2 - a_1^2)^3}{4(3a_2 - a_1^2)^3 + (27a_3 - 9a_1a_2 + 2a_1^3)^2}.$$

3-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Kornev, 2025)

On $\mathbb{CP}^1$, for each nonzero $c \in \mathbb{C}$ there exists a structure (which we call *equiharmonic*) of an algebraic 3-valued coset group $\mathbb{G}_{3,\text{eqh}}(\mathbb{CP}^1)$ with neutral element 0, the map $\text{inv}(x) = -x$, and addition given by the polynomial $p_{3,\text{eqh}}(z; x, y)$, where

$$\begin{aligned} p_{3,\text{eqh}}(-z; x, y) = & e_1^3 - 27e_3 \\ & + 18c e_1^2 e_3 - 54c e_2 e_3 - 27c^2 e_2^2 e_3 + 81c^2 e_1 e_3^2, \end{aligned}$$

and e_k denotes the k -th elementary symmetric function in x, y, z . All such 3-valued groups are isomorphic.

Symmetric 3-Algebraic 3-Valued Groups

Let us now describe all symmetric 3-algebraic 3-valued groups.

The general form of the polynomials P_3 is as follows:

$$\begin{aligned} &\sigma_1^3 + k_0\sigma_3 + k_6\sigma_3^2 + k_{12}\sigma_3^3 + k_2\sigma_1\sigma_3 + k_4\sigma_2\sigma_3 + \ell_6\sigma_1\sigma_2\sigma_3 \\ &+ \ell_4\sigma_1^2\sigma_3 + k_8\sigma_2^2\sigma_3 + \ell_8\sigma_3^2\sigma_1 + k_{10}\sigma_3^2\sigma_2, \end{aligned}$$

where σ_i 's are elementary symmetric polynomials and k_i 's are parameters.

Theorem (Buchstaber, Kornev, 2025)

There are two classes of symmetric 3-algebraic 3-valued groups:

(I) If $k_0 = -27$, we have

$$P_3(z; -x, -y) = (\sigma_1^3 - 27\sigma_3) + 18c\sigma_1^2\sigma_3 - 54c\sigma_2\sigma_3 - 27c^2\sigma_2^2\sigma_3 + 81c^2\sigma_1\sigma_3^2.$$

(II) If $k_0 = 0$, we have

$$P_3(z; -x, -y) = (\sigma_1 + \alpha\sigma_3)^3,$$

In this case, the 3-valued group is obtained by the diagonal construction $G \times G \xrightarrow{\mu} G \xrightarrow{\text{Sym}^3 \circ \text{diag}} \text{Sym}^3(G)$ from the 1-algebraic 1-valued group G , constructed by the polynomial $P_1(z; -x, -y) = \sigma_1 + \alpha\sigma_3$.

4-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Kornev, 2025)

On $\mathbb{CP}^1$, for each nonzero $b \in \mathbb{C}$ there exists a structure (which we call harmonic) of an involutive algebraic 4-valued coset group $\mathbb{G}_{4,\text{har}}(\mathbb{CP}^1)$ with neutral element 0, whose addition is given by the polynomial

$$\begin{aligned} p_{4,\text{har}}(z; x, y) = & e_1^4 - 8 e_1^2 e_2 + 16 e_2^2 - 128 e_1 e_3 \\ & - 112 b e_1^2 e_3 - 4 b^2 e_1^3 e_3 - 64 b e_2 e_3 \\ & - 112 b^2 e_1 e_2 e_3 - 64 b^2 e_3^2 \\ & - 288 b^3 e_1 e_3^2 + 6 b^4 e_1^2 e_3^2 - 136 b^4 e_2 e_3^2 \\ & - 112 b^5 e_3^3 - 4 b^6 e_1 e_3^3 + b^8 e_3^4 \end{aligned}$$

where e_k denotes the k -th elementary symmetric function in x, y, z . All such 4-valued groups are isomorphic.

Theorem (Buchstaber, Kornev, 2025)

For each nonzero $c \in \mathbb{C}$, there exists a unique structure (*equiharmonic*) of an involutive 6-valued algebraic coset group $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$ with neutral element 0 and addition given by

$$\begin{aligned} p_{6,\text{eqh}}(z; -x, -y) = & e_1^6 - 2^2 \cdot 3 e_1^4 e_2 + 2^4 \cdot 3 e_1^2 e_2^2 - 2^6 e_2^3 - 2 \cdot 3^4 \cdot 17 e_1^3 e_3 \\ & - 2^3 \cdot 3^4 \cdot 19 e_1 e_2 e_3 + 3^3 \cdot 19^3 e_3^2 \\ & - 2^5 \cdot 3^2 \cdot 11 c e_1^4 e_3 - 2^2 \cdot 3^3 \cdot 5 c^2 e_1^5 e_3 - 2^4 \cdot 3^2 \cdot 211 c e_1^2 e_2 e_3 \\ & - 2^2 \cdot 3^3 \cdot 197 c^2 e_1^3 e_2 e_3 - 2^3 \cdot 3^5 c^3 e_1^4 e_2 e_3 - 2^6 \cdot 3^2 \cdot 5^2 c e_2^2 e_3 \\ & - 2^4 \cdot 3^3 \cdot 107 c^2 e_1 e_2^2 e_3 - 2^4 \cdot 3^7 c^3 e_1^2 e_2^2 e_3 - 2 \cdot 3^6 c^4 e_1^3 e_2^2 e_3 \\ & - 2^6 \cdot 3^5 c^3 e_2^3 e_3 - 2^3 \cdot 3^7 c^4 e_1 e_2^3 e_3 + 2^4 \cdot 3^3 \cdot 7^2 \cdot 19 c e_1 e_3^2 \\ & + 2^2 \cdot 3^3 \cdot 47 \cdot 53 c^2 e_1^2 e_3^2 + 2^3 \cdot 3^5 \cdot 61 c^3 e_1^3 e_3^2 + 2 \cdot 3^7 \cdot 17 c^4 e_1^4 e_3^2 \\ & + 2^2 \cdot 3^4 \cdot 701 c^2 e_2 e_3^2 + 2^3 \cdot 3^6 \cdot 7 \cdot 13 c^3 e_1 e_2 e_3^2 + 2^{10} \cdot 3^6 c^4 e_1^2 e_2 e_3^2 \\ & + 2^4 \cdot 3^8 \cdot 5 c^5 e_1^3 e_2 e_3^2 + 2 \cdot 3^7 \cdot 5 \cdot 17 c^4 e_2^2 e_3^2 + 2^5 \cdot 3^8 \cdot 7 c^5 e_1 e_2^2 e_3^2 \\ & + 2^2 \cdot 3^9 \cdot 17 c^6 e_1^2 e_2^2 e_3^2 + 2^2 \cdot 3^9 \cdot 11 c^6 e_2^3 e_3^2 + 2^3 \cdot 3^{11} c^7 e_1 e_2^3 e_3^2 \\ & + 3^{12} c^8 e_2^4 e_3^2 - 2^3 \cdot 3^{12} c^3 e_3^3 - 2 \cdot 3^9 \cdot 5 \cdot 47 c^4 e_1 e_3^3 \\ & - 2^4 \cdot 3^9 \cdot 17 c^5 e_1^2 e_3^3 - 2^2 \cdot 3^9 \cdot 5 c^6 e_1^3 e_3^3 - 2^6 \cdot 3^{11} c^5 e_2 e_3^3 \\ & - 2^2 \cdot 3^{10} \cdot 5 \cdot 11 c^6 e_1 e_2 e_3^3 - 2^3 \cdot 3^{11} c^7 e_1^2 e_2 e_3^3 - 2^3 \cdot 3^{11} \cdot 5 c^7 e_2^2 e_3^3 \\ & - 2 \cdot 3^{12} c^8 e_1 e_2^2 e_3^3 - 2^3 \cdot 3^{12} c^6 e_3^4 - 2^3 \cdot 3^{12} c^7 e_1 e_3^4 \\ & + 3^{12} c^8 e_1^2 e_3^4 - 2^2 \cdot 3^{13} c^8 e_2 e_3^4. \end{aligned}$$

Polynomials $p_d(z; x, y)$

Introduce the following symmetric polynomials:

$$p_d(z; x, y) = \prod_{r,s=1}^d \left(\sqrt[d]{z} + \varepsilon^r \sqrt[d]{x} + \varepsilon^s \sqrt[d]{y} \right),$$

where $\varepsilon = e^{2\pi i/d}$ and $\sqrt[d]{-}$ denotes some branch of the root.

$$p_1 = \sigma_1,$$

$$p_2 = \sigma_1^2 - 4\sigma_2,$$

$$p_3 = \sigma_1^3 - 27\sigma_3,$$

$$p_4 = \sigma_1^4 - 2^3 \sigma_1^2 \sigma_2 + 2^4 \sigma_2^2 - 2^7 \sigma_1 \sigma_3,$$

$$p_5 = \sigma_1^5 - 5^4 \sigma_1^2 \sigma_3 - 5^5 \sigma_2 \sigma_3,$$

$$p_6 = \sigma_1^6 - 2^2 \cdot 3 \sigma_1^4 \sigma_2 - 2 \cdot 3^4 \cdot 17 \sigma_1^3 \sigma_3 + 2^4 \cdot 3 \sigma_1^2 \sigma_2^2 \\ - 2^3 \cdot 3^4 \cdot 19 \sigma_1 \sigma_2 \sigma_3 - 2^6 \sigma_2^3 + 3^3 \cdot 19^3 \sigma_3^2$$

where σ_j 's are elementary symmetric polynomials in x, y, z .

Groups $\mathbb{G}_d(\mathbb{C})$

The polynomial

$$p_d(z; (-1)^d x, (-1)^d y)$$

defines a commutative algebraic d -valued group $\mathbb{G}_d(\mathbb{C})$ on $\mathbb{C}$ with multiplication

$$x * y = [z \mid p_d(z; x, y) = 0],$$

neutral element 0 and inverse $\text{inv}(x) = (-1)^d x$.

θ -Circulants

For any elements $a_0, \dots, a_{n-1} \in \mathbb{k}$, introduce a θ -circulant matrix:

$$\text{Circ}_{\theta}(a_0, \dots, a_{n-1}) = \begin{pmatrix} a_0 & \theta a_1 & \theta a_2 & \cdots & \theta a_{n-1} \\ a_{n-1} & a_0 & \theta a_1 & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \theta a_2 \\ \vdots & \ddots & \ddots & \ddots & \theta a_1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

Wendt Matrices

Theorem (Buchstaber, Kornev, 2025)

The polynomial $p_d(z; x, y)$ defining the d -valued multiplication is the determinant of a y -circulant $d \times d$ matrix:

$$\text{Circ}_y \left(z + (-1)^{d+1}x + y, \binom{d}{1}w, \dots, \binom{d}{n-1}w^{d-1} \right)$$

where $w^d = z$.

These matrices generalize the Wendt matrices (the substitution $x = (-1)^d$, $y = z = 1$).

$p_d(z; x, y)$ and Wendt Matrices

$$p_2 = \begin{vmatrix} z - x + y & 2wy \\ 2w & z - x + y \end{vmatrix}$$

$$p_3 = \begin{vmatrix} z + x + y & 3yw & 3yw^2 \\ 3w^2 & z + x + y & 3yw \\ 3w & 3w^2 & z + x + y \end{vmatrix}$$

$$p_4 = \begin{vmatrix} z - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & z - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & z - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & z - x + y \end{vmatrix}$$

Polynomials $p_d(z; x, y)$

Theorem (Buchstaber, Kornev, 2025)

For prime $d \geq 5$, the polynomial

$$p_d(z; x, y) - (x + y + z)^d$$

is divisible by d^4xyz .

This result follows from the above observations and from the following:

Theorem (Wolstenholme, [Wol62])

$$\binom{2d-1}{d-1} - 1$$

is divisible by d^3 for primes $d \geq 5$.

Fermat Curves and $\mathbb{G}_d(\mathbb{C})$

Theorem (Buchstaber, Kornev, 2025)

Consider the Fermat curve ($d \geq 2$)

$$M_d^1 = \{x^d + y^d = z^d\}.$$

Then the dual curve $(M_d^1)^\vee \subset (\mathbb{P}^2)^*$ is given by the equation

$$p_{d-1}(w^d; u^d, v^d) = 0.$$

Algebraic Monoids $\mathbb{M}_d(\mathbb{C}P^1)$

Theorem (Buchstaber, Kornev, 2025)

The group $\mathbb{G}_d(\mathbb{C})$ extends (only) to an algebraic d -valued coset monoid $\mathbb{M}_d(\mathbb{C}P^1)$ on $\mathbb{C}P^1$ with

$$* : \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$$

$$(x_1 : x_0) * (y_1 : y_0) = (b_d : b_{d-1} : \cdots : b_0),$$

where $b_j = b_j(x, y)$ is the coefficient of $z_1^{d-j} z_0^j$ in the homogeneous polynomial

$$(x_0 y_0 z_0)^d p_d \left(\frac{z_1}{z_0}; (-1)^d \frac{x_1}{x_0}, (-1)^d \frac{y_1}{y_0} \right)$$

whenever $(x_1 : x_0)$ and $(y_1 : y_0)$ are not **both** equal to $(1 : 0)$.

Algebraic Monoids $\mathbb{M}_d(\mathbb{C}P^1)$

Here the point ∞ is absorbing, i.e.

$$\infty * x = x * \infty = [\infty, \infty, \dots, \infty]$$

for any $x \in \mathbb{C}P^1 \setminus \{\infty\}$, and the value $\infty * \infty$ is undefined.

Projective Duality is a Shift on $\mathbb{M}_d$'s

For each $d \geq 2$ consider a curve

$$X_d = \{p_d(z; x, y) = 0\}.$$

Theorem (Buchstaber, Kornev, 2025)

Under projective duality the curve X_d ($d \geq 2$) goes to

$$X_d^\vee = \{ (uvw)^{d-1} p_{d-1}(1/w; 1/u, 1/v) = 0 \} \subset (\mathbb{CP}^2)^*.$$

The composition of the duality $X_d \mapsto X_d^\vee$ with the subsequent Möbius transformation $(u, v, w) \mapsto (1/u, 1/v, 1/w)$ defines a shift operation

$$\mathbb{M}_d(\mathbb{CP}^1) \mapsto \mathbb{M}_{d-1}(\mathbb{CP}^1)$$

in the family of algebraic d -valued monoids.

Projective Duality is a Shift on $\mathbb{M}_d$'s

Example. For $X_2^\vee$ we have the parametrization

$$(u, v) = \left(-\frac{1}{1+t}, \frac{1}{t} \right) \quad \text{or} \quad \frac{1}{u} + \frac{1}{v} = -1.$$

Taking the projective closure (homogenization), we find that $X_2^\vee$ is given by the zero locus of the polynomial

$$P_1 = uvw p_1(w^{-1}; u^{-1}, v^{-1}) = (u + v)w + uv.$$

Projective Duality is a Shift on $\mathbb{M}_d$'s

Example. For $X_3^\vee$:

$$(u, v) = \left(\frac{1}{(1+t)^2}, \frac{1}{t^2} \right) \quad \text{or} \quad \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} = 1.$$

The curve $X_3^\vee$ is given by the polynomial

$$P_2 = (uvw)^2 p_2(w^{-1}; u^{-1}, v^{-1}) = (uv - w(u + v))^2 - 4uvw^2,$$

$$P_2 = (uv)^2 + (vw)^2 + (uw)^2 - 2u^2vw - 2uv^2w - 2uvw^2.$$

Projective Duality is a Shift on $\mathbb{M}_d$'s

For any smooth curve $X \subset \mathbb{P}^2$ of degree d , according to Plücker formulas [GKZ94, Proposition 2.4], we have $\deg X^\vee = d(d-1)$.

Since $\deg X_d^\vee = (d-1)^2$ for $d \geq 3$, the curve X_d cannot be smooth.

The curve X_2 is smooth.

Discriminants and X_d 's

Theorem (Gaiur, Rubtsov, van Straten, 2024)

The discriminant $\Delta_t(P)$ of the polynomial

$$P(t) = (zt^{d-1} + y)(1 + t)^{d-1} + (-1)^{d-1}xt^{d-1}$$

with respect to the variable t , which is a polynomial of degree $4d - 6$, is related to $p_d(z; x, y)$ by

$$(-1)^d(d-1)^{2d-2}(xyz)^{d-2} p_d(z; x, y) = \Delta_t(P)$$

for each $d \geq 2$.

This result has a nice explanation from the point of view of projective duality.

POV: Projective Duality

Consider the curve $X_d^\vee$. Its parametrization in the chart $\{w = 1\}$:

$$(u, v) = ((-1 - t)^{1-d}, t^{1-d}).$$

By the definition of the $X_d^\vee$ -discriminant, the curve $X_d^{\vee\vee} = \{p_d(z; x, y) = 0\}$ is an irreducible component of the discriminant of the polynomial of t obtained by restricting the line

$$ux + vy + 1 = 0$$

onto $X_d^\vee$, i.e. the curve $X_d^{\vee\vee}$ is the discriminant in t of $P(t)$, where $P(t)$ is a polynomial obtained from the equation

$$\frac{1}{(-1 - t)^{d-1}} \cdot x + \frac{1}{t^{d-1}} \cdot y + 1 = 0$$

after homogenization.

POV: Projective Duality

If $xyz = 0$, then for $d \geq 2$ the polynomial $P(t)$ has a $(d - 1)$ -fold root $t^* \in \{0, -1, \infty\}$. Thus, $\Delta_t(P)$ is divisible by a certain power of the monomial xyz .

By [GRS24, Theorem 2.2], $\Delta_t(P)$ has no other singular components. The required statement now follows by comparing degrees.

Polynomials $p_{d,n}$

In connection with Bessel kernels for solutions of Picard–Fuchs differential equations for the kernel

$$K_d = \sum_{j,k} \binom{j+k}{k}^d \frac{x^j y^k}{z^{j+k}},$$

the iterated analogue of the polynomials $p_d(z; x, y)$ was considered in [GRS24]:

$$p_{d,n}(z; \mathbf{x}) = \prod_{k_1, \dots, k_n=1}^d \left(\sqrt[d]{z} + \varepsilon^{k_1} \sqrt[d]{x_1} + \dots + \varepsilon^{k_n} \sqrt[d]{x_n} \right).$$

Operations $O_{d,n}(\mathbb{C}P^1)$

The polynomial $p_{d,n}(z; \mathbf{x})$ defines an n -ary d^{n-1} -valued algebraic operation

$$\mu(x_1, \dots, x_n) = [z \mid p_{d,n}(z; \mathbf{x}) = 0].$$

Denote by $O_{d,n}(\mathbb{C}P^1)$ the variety $\mathbb{C}P^1$ with the operation μ .

Let

$$X_d^{n-1} = \{p_{d,n} = 0\}$$

be the hypersurface in $\mathbb{C}P^n$. For integers $d \geq 2$ and $n \geq 2$ define

$$P_{d,n} = (u_1 \cdots u_n w)^{d-1} p_{d-1}(w^{-1}; u_1^{-1}, \dots, u_n^{-1})$$

the polynomial of degree d^{n-1} .

Operations $O_{d,n}(\mathbb{C}P^1)$

Theorem (Buchstaber, Kornev, 2025)

The composition of the duality ($d \geq 2, n \geq 2$)

$$X_d^{n-1} \mapsto (X_d^{n-1})^\vee = \{P_{d,n} = 0\} \subset (\mathbb{C}P^n)^*$$

with the subsequent Möbius transformation

$$(u_1, \dots, u_n, w) \mapsto (1/u_1, \dots, 1/u_n, 1/w)$$

defines a shift operation

$$O_{d,n}(\mathbb{C}P^1) \mapsto O_{d-1,n}(\mathbb{C}P^1)$$

in the family of n -ary d^{n-1} -valued algebraic structures $O_{d,n}(\mathbb{C}P^1)$.

Fermat Hypersurfaces and $p_{d,n}$'s

Theorem (Buchstaber, Kornev, 2025)

Let M_d^{n-1} be a Fermat hypersurface

$$M_d^{n-1} = \{x_1^d + x_2^d + \dots + x_n^d = z^d\}$$

in $\mathbb{C}P^n$ with coordinates $x_1, \dots, x_n, z$. The dual hypersurface is defined by the equation

$$(M_d^{n-1})^\vee = \{p_{d-1,n}(w^d; u_1^d, \dots, u_n^d) = 0\}$$

in $(\mathbb{C}P^n)^*$ with the dual coordinates $u_1, \dots, u_n, w$.

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Thank you for attention!