

n -Algebraic Monoids, Cubics, and Discriminants*

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Algebraic n -Valued Monoids and Groups

An *algebraic n -valued monoid* is an algebraic variety X equipped with an associative n -valued multiplication given by a *rational morphism* $X \times X \rightarrow \text{Sym}^n(X)$ with a neutral element $e \in X$ such that

$$x * e = e * x = [x, x, \dots, x] \quad \text{for every } x \in X.$$

An *algebraic n -valued group* is an algebraic n -valued monoid on X together with a *regular morphism* $\text{inv} : X \rightarrow X$ such that for any $x \in X$ the following two conditions hold:

$$e \in x * \text{inv}(x), \quad x * \text{inv}(x) = \text{inv}(x) * x.$$

Hadamard monoid on $\mathbb{CP}^1$

Consider $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

There is a structure of a 1-valued commutative algebraic monoid $M_{\text{mult}}(\mathbb{CP}^1)$ with identity element $1 = [1 : 1]$ and multiplication

$$(z_1 : z_0) \cdot (w_1 : w_0) = (z_1 w_1 : z_0 w_0),$$

defined on $(\text{Sym}^2 \mathbb{CP}^1) \setminus \{[0, \infty]\}$.

The element $\infty = (1 : 0)$ is absorbing in this monoid, i.e. $z * \infty = \infty$ for each $z \in \mathbb{CP}^1 \setminus \{0\}$.

The elements of $\mathbb{CP}^1 \setminus \{0, \infty\}$, and only they, have inverse $\text{inv}(z_1 : z_0) = (z_0 : z_1)$.

Monoid $M_{\text{cusp}}(\mathbb{CP}^1)$

On $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ there is also a structure $M_{\text{cusp}}(\mathbb{CP}^1)$ of a 1-valued commutative algebraic monoid with neutral element 0 and addition

$$(z_1 : z_0) \cdot (w_1 : w_0) = (z_1 w_0 + z_0 w_1 : z_0 w_0),$$

defined on $(\text{Sym}^2 \mathbb{CP}^1) \setminus \{[\infty, \infty]\}$.

The elements of $\mathbb{CP}^1 \setminus \{\infty\}$, and only they, have inverse $\text{inv}(z_1 : z_0) = (-z_1 : z_0)$.

Consider a polynomial

$$P(z; x, y) = -x - y + z + \alpha xyz,$$

where $\alpha \in \mathbb{C}$. Then we have a 1-algebraic 1-valued commutative group on $\mathbb{C}$ with the operation

$$x * y = \frac{x + y}{1 + \alpha xy}.$$

This algebraic abelian group is isomorphic to the algebraic 1-valued group of matrices of the form

$$\begin{pmatrix} 1 & x \\ \alpha x & 1 \end{pmatrix}$$

modulo multiplication by nonzero complex numbers, since the following identity holds:

$$\begin{pmatrix} 1 & x \\ \alpha x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ \alpha y & 1 \end{pmatrix} = (1 + \alpha xy) \begin{pmatrix} 1 & \frac{x+y}{1+\alpha xy} \\ \alpha \frac{x+y}{1+\alpha xy} & 1 \end{pmatrix}.$$

Etale Construction

Let M be a single-valued monoid on a set X , and let H be a subgroup of order n of its automorphism group $\text{Aut}(M)$.

We call the *coset n -valued monoid* M_H the result of applying the construction of [Buc06, Theorem 1] to M and H :

Let $\pi : M \rightarrow X = M/H$, suppose $\pi(g_1) = x_1$ and $\pi(g_2) = x_2$.

$$x_1 * x_2 = [\pi(g_1 \cdot \zeta(g_2)) \mid \zeta \in H].$$

Proposition

The notion of a coset monoid is well-defined.

Polynomials $p_d(z; x, y)$

Introduce the following symmetric integer polynomials:

$$p_d(z; x, y) = \prod_{r,s=1}^d \left(\sqrt[d]{z} + \varepsilon^r \sqrt[d]{x} + \varepsilon^s \sqrt[d]{y} \right),$$

where $\varepsilon = e^{2\pi i/d}$ and $\sqrt[d]{-}$ denotes some branch of the root.

$$p_1 = \sigma_1,$$

$$p_2 = \sigma_1^2 - 4\sigma_2,$$

$$p_3 = \sigma_1^3 - 27\sigma_3,$$

$$p_4 = \sigma_1^4 - 2^3 \sigma_1^2 \sigma_2 + 2^4 \sigma_2^2 - 2^7 \sigma_1 \sigma_3,$$

$$p_5 = \sigma_1^5 - 5^4 \sigma_1^2 \sigma_3 - 5^5 \sigma_2 \sigma_3,$$

$$p_6 = \sigma_1^6 - 2^2 \cdot 3 \sigma_1^4 \sigma_2 - 2 \cdot 3^4 \cdot 17 \sigma_1^3 \sigma_3 + 2^4 \cdot 3 \sigma_1^2 \sigma_2^2 \\ - 2^3 \cdot 3^4 \cdot 19 \sigma_1 \sigma_2 \sigma_3 - 2^6 \sigma_2^3 + 3^3 \cdot 19^3 \sigma_3^2$$

where σ_j 's are elementary symmetric polynomials in x, y, z .

$p_d(z; x, y)$ and Wendt Matrices

$$p_2 = \begin{vmatrix} w^2 - x + y & 2wy \\ 2w & w^2 - x + y \end{vmatrix}$$

$$p_3 = \begin{vmatrix} w^3 + x + y & 3yw & 3yw^2 \\ 3w^2 & w^3 + x + y & 3yw \\ 3w & 3w^2 & w^3 + x + y \end{vmatrix}$$

$$p_4 = \begin{vmatrix} w^4 - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & w^4 - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & w^4 - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & w^4 - x + y \end{vmatrix}$$

θ -Circulants

For any elements $a_0, \dots, a_{n-1} \in \mathbb{k}$, introduce a θ -circulant matrix:

$$\text{Circ}_{\theta}(a_0, \dots, a_{n-1}) = \begin{pmatrix} a_0 & \theta a_1 & \theta a_2 & \cdots & \theta a_{n-1} \\ a_{n-1} & a_0 & \theta a_1 & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \theta a_2 \\ \vdots & \ddots & \ddots & \ddots & \theta a_1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

Wendt Matrices

Theorem (Buchstaber, Kornev, 2025)

The polynomial $p_d(z; x, y)$ defining the d -valued multiplication is the determinant of a y -circulant $d \times d$ matrix:

$$\text{Circ}_y \left(w^d + (-1)^{d+1}x + y, \binom{d}{1}w, \dots, \binom{d}{n-1}w^{d-1} \right)$$

where $w^d = z$.

These matrices generalize the Wendt matrices known in the context of the Fermat Last Theorem (for the substitution $x = (-1)^d$, $y = z = 1$).

Polynomials $p_d(z; x, y)$

Theorem (Buchstaber, Kornev, 2025)

For prime $d \geq 5$, the polynomial

$$p_d(z; x, y) - (x + y + z)^d$$

is divisible by d^4xyz .

This result follows from the above observations and from the following:

Theorem (Wolstenholme, [Wol62])

$$\binom{2d-1}{d-1} - 1$$

is divisible by d^3 for primes $d \geq 5$.

Groups $\mathbb{G}_d(\mathbb{C})$

The polynomial

$$p_d(z; (-1)^d x, (-1)^d y)$$

defines a commutative algebraic d -valued group $\mathbb{G}_d(\mathbb{C})$ on $\mathbb{C}$ with multiplication

$$x * y = [z \mid p_d(z; x, y) = 0],$$

neutral element 0 and inverse $\text{inv}(x) = (-1)^d x$.

Theorem (Buchstaber, Kornev, 2025)

The structure of the group $\mathbb{G}_n(\mathbb{C})$ extends (only) to the structure of an algebraic n -valued coset monoid $\mathbb{M}_n(\mathbb{C}P^1)$ on $\mathbb{C}P^1$. Here the point ∞ is absorbing, i.e. $\infty * x = x * \infty = [\infty, \infty, \dots, \infty]$ for any $x \in \mathbb{C}P^1 \setminus \{\infty\}$, and the value $\infty * \infty$ is undefined.

$$* : \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$$

$$(x_1 : x_0) * (y_1 : y_0) = (b_n : b_1 : \dots : b_0),$$

where $b_j = b_j(x, y)$ is the coefficient of $z_1^{n-j} z_0^j$ in the homogeneous polynomial

$$(x_0 y_0 z_0)^n p_n \left(\frac{z_1}{z_0}; (-1)^n \frac{x_1}{x_0}, (-1)^n \frac{y_1}{y_0} \right)$$

whenever $(x_1 : x_0)$ and $(y_1 : y_0)$ are not both equal to $(1 : 0) =: \infty$.

Isomorphisms

Two algebraic (analytic, topological) n -valued monoids (groups) X and Y are called *isomorphic* if there exists an isomorphism (homeomorphism) $\varphi : X \rightarrow Y$ inducing the commutative diagram

$$\begin{array}{ccc} X \times X & \longrightarrow & \mathrm{Sym}^2(X) \\ \downarrow \cong & & \downarrow \cong \\ Y \times Y & \longrightarrow & \mathrm{Sym}^2(Y) \end{array}$$

For example, the algebraic 1-valued monoids $M_{\mathrm{mult}}(\mathbb{C}P^1)$ and $M_{\mathrm{cusp}}(\mathbb{C}P^1)$ are *not* isomorphic, since $M_{\mathrm{mult}}(\mathbb{C}P^1)$ has elements of finite order.

Let M and N be 1-valued algebraic (topological) monoids, and let $A \subset \text{Aut}(M)$ and $B \subset \text{Aut}(N)$ be finite subgroups of order n . Say that two coset monoids M_A and N_B are isomorphic if there exist isomorphisms $\tilde{\varphi} : M \rightarrow N$ and $\varphi : M/A \rightarrow N/B$, s.t.:

$$\begin{array}{ccccc}
 & M \times M & \xrightarrow{\quad} & \text{Sym}^n(M) & \\
 & \swarrow \tilde{\varphi} \times \tilde{\varphi} & & \swarrow \cong & \\
 N \times N & \xrightarrow{\quad} & \text{Sym}^n(N) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & M/A \times M/A & \xrightarrow{\quad} & \text{Sym}^n(M/A) & \\
 & \swarrow \varphi \times \varphi & & \swarrow \cong & \\
 N/B \times N/B & \xrightarrow{\quad} & \text{Sym}^n(N/B) & &
 \end{array}$$

Moreover, all arrows connecting the front and back faces of the parallelepiped must be isomorphisms.

Example of Isomorphism

Consider the discrete coset group from [Buc06, Section 4] on the set $\mathbb{Z}_+$ of nonnegative integers with zero 0 and addition

$$x_1 * x_2 = [x_1 + x_2, |x_1 - x_2|].$$

Compare it with the coset subgroup $\mathbb{G}_2(\mathbb{Z}_+ \cdot \mathbb{Z}_+) \subset \mathbb{G}_2(\mathbb{C})$, which has zero 0 and addition

$$y_1 * y_2 = [(\sqrt{y_1} + \sqrt{y_2})^2, (\sqrt{y_1} - \sqrt{y_2})^2]$$

for all nonnegative integers y_1 and y_2 . The squaring map $x \mapsto x^2$ is a bijection between the respective orbit spaces and makes the square commute. Therefore, these two 2-valued groups are isomorphic.

Regular and n -Algebraic n -Valued Monoids

We say that an algebraic n -valued monoid (or group) on X is *regular* if the n -valued multiplication $X \times X \rightarrow \text{Sym}^n(X)$ is defined on all of $X \times X$.

A *symmetric n -algebraic n -valued monoid (group)* on $\mathbb{C}$ is an algebraic n -valued monoid (group) $\mathbb{G}_{\mathbb{C}}(f(z; x, y))$ whose (partially defined) multiplication

$$x * y = [z \mid f(z; x, y) = 0]$$

is given by a symmetric polynomial $f(z; x, y)$ in which each variable appears with degree at most n .

Buchstaber Polynomials

Consider a 3-parametrized family of polynomials:

$$B_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz)$$

In elementary symmetric functions:

$$B_a(z; x, y) = (e_1^2 - 4e_2) - 4a_1e_3 - 2a_2e_1e_3 - 4a_3e_2e_3 + (a_2^2 - 4a_1a_3)e_3^2.$$

The polynomials $B_a(z; x, y)$ endow $\mathbb{C}$ with the structure of the universal regular algebraic 2-valued group $\mathbb{G}_{\mathbb{C}}(B_a)$ for any $\mathbf{a} \in \mathbb{C}^3$, with addition

$$x * y = [z \mid B_a(z; x, y) = 0],$$

neutral element 0, and inversion map $\text{inv}(x) = x$ [BV19, Theorem 6.3], [BK25b, Theorem 1].

The Chebyshev Coset Monoid

On $M_{\text{mult}}(\mathbb{CP}^1)$ consider the involution $\tau : z \mapsto 1/z$ for each $z \in \mathbb{CP}^1$.

Points of the orbit space of the involution τ are represented by fibers of the branched double covering

$$\begin{aligned}\pi : \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1 \\ z &\mapsto \frac{1}{2}(z + 1/z)\end{aligned}$$

with branch points ± 1 .

The corresponding coset monoid $\mathbb{M}_{\text{mult}}(\mathbb{CP}^1) := M_{\text{mult}}(\mathbb{CP}^1)_{\langle \tau \rangle}$ has identity 1 and multiplication

$$x * y = \left[xy \pm \sqrt{(x^2 - 1)(y^2 - 1)} \right],$$

defined on $(\text{Sym}^2 \mathbb{CP}^1) \setminus [\infty, \infty]$.

The Chebyshev Coset Monoid

This structure of a 2-valued algebraic monoid does not extend to a structure of a 2-valued algebraic group.

The addition law is given by the roots in z of the polynomial

$$P_{\text{mult}}(z; x, y) = z^2 - 2xy z + x^2 + y^2 - 1.$$

With the multiplication in homogeneous coordinates:

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow \text{Sym}^2(\mathbb{C}P^1) \cong \mathbb{C}P^2$$

$$(x_1 : x_0) * (y_1 : y_0) = (x_1^2 y_0^2 + y_1^2 x_0^2 - x_0^2 y_0^2 : -2x_1 y_1 x_0 y_0 : x_0^2 y_0^2).$$

The Chebyshev Coset Monoid

For $x = \cos \alpha$, $y = \cos \beta$, the addition becomes the cosine addition formulas. Let

$$T_j = T_j(\cos \alpha) = \cos j\alpha$$

be the classical Chebyshev polynomials of the first kind ($j \geq 0$). Then

$$T_j * T_k = [T_{j+k}, T_{|j-k|}].$$

Consider the 2-valued submonoid $\mathbb{T} = \mathbb{T}(\mathbb{C})$ of $\mathbb{M}_{\text{mult}}(\mathbb{C}P^1)$ generated by taking integral nonnegative powers of the element $\cos \alpha$.

We have an isomorphism $\mathbb{T} \cong \mathbb{G}_2(\mathbb{Z}_+ \cdot \mathbb{Z}_+)$.

Weierstrass Form of a Cubic

Let $\delta_a = \Delta_t(t^3 + a_1 t^2 + a_2 t + a_3)$ be the discriminant. Then

$$\delta_a = -4a_3a_1^3 + a_2^2a_1^2 + 18a_2a_3a_1 - 4a_2^3 - 27a_3^2.$$

Let $\mathcal{E}$ be an irreducible cubic over the field $\mathbb{C}$. As is well known (see, e.g., [FW69, Exercise 5–24]), $\mathcal{E}$ is isomorphic to a cubic given in $\mathbb{C}P^2$ with coordinates $(x : y : z)$, in the chart $\{z = 1\}$, by

$$\mathcal{E} = \{y^2 = x^3 + a_1x^2 + a_2x + a_3\}.$$

As an abelian variety over $\mathbb{C}$, a cubic admits only automorphisms of orders 2, 3, 4, and 6 [Har77, Corollary 4.7].

Let complex parameters α, g_2, g_3 be such that

$$y^2 = (x + \alpha)^3 - \frac{g_2}{4}(x + \alpha) - \frac{g_3}{4}, \quad \begin{cases} a_1 = 3\alpha, \\ a_2 = 3\alpha^2 - \frac{g_2}{4}, \\ a_3 = \alpha^3 - \frac{g_2\alpha}{4} - \frac{g_3}{4}. \end{cases}$$

Recall that the group law

$$(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3)$$

on $\mathcal{E}$ is given (for distinct points of $\mathcal{E}$) by

$$\begin{cases} x_3 = -x_1 - x_2 - 3\alpha + \left(\frac{y_1 - y_2}{x_1 - x_2} \right)^2, \\ y_3 = (x_1 - x_3) \cdot \frac{y_1 - y_2}{x_1 - x_2} - y_1. \end{cases} \quad (1)$$

In the coincident case, (1) is understood via the limit as $x_2 \rightarrow x_1$.

2-Valued Structures on $\mathbb{CP}^1$

There is a branched double covering

$$\begin{aligned}\pi : \mathcal{E} &\rightarrow \mathbb{CP}^1, \\ (x, y) &\mapsto x, \\ \infty &\mapsto \infty.\end{aligned}$$

with branch points at $\{x^3 + a_1x^2 + a_2x + a_3 = 0\}$ and at ∞ . The fibers of π are in bijection with the points of the orbit space $\mathcal{E}/\langle\sigma\rangle$ for the involution

$$\begin{aligned}\sigma : (x, y) &\mapsto (x, -y), \\ \infty &\mapsto \infty\end{aligned}$$

Applying the coset construction, we obtain a structure $\mathcal{E}_{\langle\sigma\rangle}$ of a coset algebraic 2-valued group on $\mathbb{CP}^1$ with neutral element ∞ :

$$x_1 * x_2 = \left[-x_1 - x_2 - 3\alpha + \left(\frac{y_1 \pm y_2}{x_1 - x_2} \right)^2 \right].$$

2-Valued Structures on $\mathbb{CP}^1$

Proposition

The values of $x_1 * x_2$ are the roots of a quadratic polynomial $D(z; x_1, x_2)$ in z :

$$D(z; x_1, x_2) = \Theta_0(x_1, x_2) z^2 + \Theta_1(x_1, x_2) z + \Theta_2(x_1, x_2),$$

$$\left\{ \begin{array}{l} \Theta_0 = 16(x_1 - x_2)^2, \\ \Theta_1 = 8(2g_3 + g_2(x_1 + x_2 + 2\alpha) - \\ \quad - 4(x_1 x_2(x_1 + x_2) + 6x_1 x_2 \alpha + 3(x_1 + x_2)\alpha^2 + 2\alpha^3)), \\ \Theta_2 = (g_2 + 4x_1 x_2)^2 + 16g_2(x_1 + x_2)\alpha + 24(g_2 - 4x_1 x_2)\alpha^2 \\ \quad - 64(x_1 + x_2)\alpha^3 - 48\alpha^4 + 16g_3(x_1 + x_2 + 3\alpha). \end{array} \right.$$

Generalized Kontsevich Polynomial

In [BGR24], $D(z; x_1, x_2)$ was called the *generalized Kontsevich polynomial*. The first appearance of these polynomials, is due to the explicit computations in Geometric Langlands Correspondence.

As noted in [BGR24], one has

$$D_a(z; x, y) = (xyz)^2 B_a \left(-\frac{1}{z}; -\frac{1}{x}, -\frac{1}{y} \right),$$

where $D_a(z; x, y)$ is the generalized Kontsevich polynomial.

2-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Gaiur, Rubtsov) [BGR24]

(I) If $\delta_a \neq 0$, the algebraic 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(D_a) \cong \mathbb{G}_{\mathbb{CP}^1}(B_a)$ with identity ∞ is the regular coset group $\mathcal{E}_{\langle\sigma\rangle}$ for the group of points of the elliptic curve $\mathcal{E}$ with respect to the involution $\sigma : (x, y) \mapsto (x, -y)$. The group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ on $\mathbb{CP}^1$ has zero $(0 : 1)$ and addition

$$\mu_a : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^2$$

$$(x_1 : x_0) * (y_1 : y_0) = (u_2 : u_1 : u_0)$$

$$\begin{cases} u_2 = (x_1 y_0 - x_0 y_1)^2, \\ -\frac{1}{2} u_1 = x_1 x_0 (2a_1 y_1 y_0 + a_2 y_1^2 + y_0^2) + x_1^2 y_1 (a_2 y_0 + 2a_3 y_1) + x_0^2 y_0 y_1, \\ u_0 = x_1^2 y_1 (a_2^2 y_1 - 4a_3(a_1 y_1 + y_0)) - 2x_0 x_1 y_1 (a_2 y_0 + 2a_3 y_1) + x_0^2 y_0^2. \end{cases}$$

2-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Kornev, 2025)

(II) The isomorphism class of the coset algebraic 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ is completely determined by the j -invariant of the elliptic curve $\mathcal{E}$:

$$j(a) = 6912 \frac{(3a_2 - a_1^2)^3}{4(3a_2 - a_1^2)^3 + (27a_3 - 9a_1a_2 + 2a_1^3)^2}.$$

Iterating Elements

An element w of an n -valued monoid M is called *iterating* if for any $m \in M$ each of the multisets $w * m$ and $m * w$ contains an element u (generally depending on m) with multiplicity at least 2.

For $n = 2$ we call an iterating element w a *doubling* element.

Proposition (Buchstaber, Kornev, 2025)

The set $\mathbb{W} = \mathbb{W}(\mathbb{M})$ of doubling elements of a 2-valued monoid (respectively, an involutive group) $\mathbb{M}$ forms a diagonal 2-valued submonoid (respectively, subgroup).

We call the 1-valued monoid (group) from the proof of Proposition 29 the *monoid (group) of doubling points*.

Iterating Elements

The Chebyshev coset 2-valued algebraic monoid has exactly *2 doubling elements at the branch points (1 and -1)* of the branched double covering

$$\begin{aligned}\pi : \mathbb{C}P^1 &\rightarrow \mathbb{C}P^1 \\ z &\mapsto \frac{1}{2}(z + 1/z)\end{aligned}$$

Indeed, let y be an iterating element. Then the polynomial

$$P_{\text{mult}}(z; x, y) = z^2 - 2xyz + x^2 + y^2 - 1$$

has a multiple root in z for any x , i.e.

$$\Delta_z(P_{\text{mult}}(z; x, y)) = 4(x^2 - 1)(y^2 - 1) = 0.$$

Hence $y \in \{\pm 1\}$. The resulting single-valued group of doubling points is isomorphic to $\mathbb{Z}/2$.

Theorem (Buchstaber, Kornev, 2025)

Let $\mathcal{E} = \{y^2 = f(x)\}$ be an elliptic curve, where $f(x) = x^3 + a_1x^2 + a_2x + a_3$. Then:

- 1 The doubling elements of the 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(B_a(z; x, y))$ are precisely the elements of the form $1/w$, where w ranges over the branch points of the branched covering

$$\pi : \mathcal{E} \rightarrow \mathbb{CP}^1$$

$$(x, y) \mapsto x,$$

$$\infty \mapsto \infty.$$

- 2 The single-valued group of doubling points of the 2-valued group $\mathbb{G}_{\mathbb{CP}^1}(B_a)$ is isomorphic to the Klein four-group $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Factorization of Discriminant

The discriminant of the polynomial $B_a(z; x, y)$ factorizes with separated variables:

$$\Delta_z(B_a(z; x, y)) = 16 x^4 f(1/x) \cdot y^4 f(1/y). \quad (2)$$

We note that an important first application of (2) was obtained by Dragović in [Dra10] (see also [Dra14]), based on a remarkable relation between the associativity equation for a 2-valued group on $\mathbb{C}$ and the integration method for the Kovalevskaya top.

The separation property in the discriminant factorization fails for the n -valued laws $p_n(z; x, y)$ already at $n = 3$:

$$\Delta_z(p_3(z; x, y)) = -3^9 x^2 (x - y)^2 y^2.$$

3-Valued Structures on $\mathbb{CP}^1$

There is a unique projective equivalence class of nonsingular cubic curves whose groups of points contain elements of order 3 (in this case the j -invariant equals 0).

For each complex number $c \neq 0$, the *equiharmonic cubic*

$$\mathcal{E} = \{ y^2 = x^3 + c \}$$

belongs to this class [Dol12, Theorem 3.1.3].

3-Valued Structures on $\mathbb{C}P^1$

Introduce the slope

$$m = m((x_1, y_1), (x_2, y_2)) = \frac{y_1 - y_2}{x_1 - x_2}.$$

Then we have for the addition law:

$$\begin{cases} x_3 = -x_1 - x_2 + m^2, \\ y_3 = m(x_1 - x_3) - y_1. \end{cases}$$

The curve $\mathcal{E}$ admits an automorphism

$$\varphi_3 : (x, y) \mapsto (\varepsilon x, y)$$

of order 3 as an abelian variety, where $\varepsilon = e^{2\pi i/3}$. Indeed,

$$\begin{aligned} \varphi_3(x_1, y_1) \oplus \varphi_3(x_2, y_2) &= \left(-\varepsilon x_1 - \varepsilon x_2 + \left(\frac{y_1 - y_2}{\varepsilon x_1 - \varepsilon x_2} \right)^2, \right. \\ &\quad \left. \frac{y_1 - y_2}{\varepsilon x_1 - \varepsilon x_2} (\varepsilon x_1 - \varepsilon x_3) - y_1 \right) = \varphi_3(x_3, y_3). \end{aligned}$$

3-Valued Structures on $\mathbb{CP}^1$

The orbit $\{(x, y), (\varepsilon x, y), (\varepsilon^2 x, y)\}$ of the automorphism φ_3 corresponds bijectively to the value of y . There is a branched triple covering

$$\pi : \mathcal{E} \rightarrow \mathbb{CP}^1$$

$$(x, y) \mapsto y,$$

$$\infty \mapsto \infty.$$

whose base is identified with the orbit space $\mathcal{E} / \langle \varphi_3 \rangle$. The branch points are $\pm\sqrt{c}$ and ∞ .

$$\begin{aligned} y_1 * y_2 &= [\pi((x_1, y_1) \oplus (\varepsilon^k x_2, y_2)) \mid k = 0, 1, 2] \\ &= \left[m_k \left(2\sqrt[3]{y_1^2 - c} + \varepsilon^k \sqrt[3]{y_2^2 - c} - m_k^2 \right) - y_1 \right], \end{aligned}$$

$$\text{where } m_k = \frac{y_1 - y_2}{\sqrt[3]{y_1^2 - c} - \varepsilon^k \sqrt[3]{y_2^2 - c}}, \quad k = 0, 1, 2.$$

3-Valued Structures on $\mathbb{CP}^1$

Theorem (Buchstaber, Kornev, 2025)

On $\mathbb{CP}^1$, for each nonzero $c \in \mathbb{C}$ there exists a structure (which we call *equiharmonic*) of an algebraic 3-valued coset group $\mathbb{G}_{3,\text{eqh}}(\mathbb{CP}^1)$ with neutral element 0, the map $\text{inv}(x) = -x$, and addition given by the polynomial $p_{3,\text{eqh}}(z; x, y)$, where

$$\begin{aligned} p_{3,\text{eqh}}(-z; x, y) = & e_1^3 - 27e_3 \\ & + 18c e_1^2 e_3 - 54c e_2 e_3 - 27c^2 e_2^2 e_3 + 81c^2 e_1 e_3^2, \end{aligned}$$

and e_k denotes the k -th elementary symmetric function in x, y, z . All such 3-valued groups are isomorphic.

Theorem (Buchstaber, Kornev, 2025)

There are two classes of symmetric 3-algebraic 3-valued groups:

(I) If $k_0 = -27$, we have

$$P_3(z; -x, -y) = (\sigma_1^3 - 27\sigma_3) + 18c\sigma_1^2\sigma_3 - 54c\sigma_2\sigma_3 - 27c^2\sigma_2^2\sigma_3 + 81c^2\sigma_1\sigma_3^2.$$

(II) If $k_0 = 0$, we have

$$P_3(z; -x, -y) = (\sigma_1 + \alpha\sigma_3)^3,$$

In this case, the 3-valued group is obtained by the diagonal construction $G \times G \xrightarrow{\mu} G \xrightarrow{\text{Sym}^3 \circ \text{diag}} \text{Sym}^3(G)$ from the 1-algebraic 1-valued group G , constructed by the polynomial $P_1(z; -x, -y) = \sigma_1 + \alpha\sigma_3$.

Proposition (Buchstaber, Kornev, 2025)

The set $\{0, \pm 1/\sqrt{c}\}$ of iterating elements of the 3-valued group $\mathbb{G}_{3,\text{eqh}}(\mathbb{CP}^1)$ is (as a 3-valued subgroup) the diagonal construction of a 1-valued group isomorphic to $\mathbb{Z}/3$.

Proof. Let y be an iterating element in $\mathbb{G}_{3,\text{eqh}}(\mathbb{CP}^1)$. Then for any $x \in \mathbb{C}$ the discriminant of the polynomial $p_{3,\text{eqh}}(z; x, y)$ vanishes:

$$-3^9 x^2 y^2 (cx^2 - 1)^2 (cy^2 - 1)^2 (x - y)^2 (9c^2 x^2 y^2 - cx^2 + 8cxy - cy^2 + 1)^2 = 0.$$

Thus precisely the elements $0, \pm 1/\sqrt{c}$ are iterating. Let $w = 1/\sqrt{c}$. We have the multiplication table

$$w * w = [-w, -w, -w],$$

$$-w * w = w * (-w) = [0, 0, 0].$$

4-Valued Structures on $\mathbb{CP}^1$

There is a unique projective equivalence class of nonsingular cubic curves whose automorphism group is isomorphic to $\mathbb{Z}/4$ (in this case the j -invariant equals 1728).

This class consists of the *harmonic cubics* [Dol12, Theorem 3.1.3]

$$\mathcal{E} = \{ y^2 = x^3 + bx \}.$$

Consider the map

$$\begin{aligned}\varphi_4 : \mathcal{E} &\rightarrow \mathcal{E} \\ (x, y) &\mapsto (-x, iy)\end{aligned}$$

which is clearly an automorphism of the abelian variety $\mathcal{E}$. The orbit space $\mathcal{E} / \langle \varphi_4 \rangle$ is identified with the fibers of

$$\begin{aligned}\pi_4 : \mathcal{E} &\rightarrow \mathbb{CP}^1 \\ (x, y) &\mapsto x^2, \\ \infty &\mapsto \infty\end{aligned}$$

Write the 4-valued addition law:

$$\begin{aligned}x_1 * x_2 &= \left[\pi_4 \left((x_1, y_1), \varphi_4^r(x_2, y_2) \right) \mid r = 0, \dots, 3 \right] \\ &= \left[\left(-\sqrt{x_1} - (-1)^\ell \sqrt{x_2} + m_{\ell,k}^2 \right)^2 \mid k, \ell = 0, 1 \right],\end{aligned}\tag{3}$$

where

$$m_{\ell,k} = \frac{\sqrt{\sqrt{x_1^3} + b\sqrt{x_1}} - (-1)^k \cdot \sqrt{(-1)^\ell (\sqrt{x_2^3} + b\sqrt{x_2})}}{\sqrt{x_1} - (-1)^\ell \sqrt{x_2}}.$$

4-Valued Structures on $\mathbb{C}P^1$

Theorem (Buchstaber, Kornev, 2025)

On $\mathbb{C}P^1$, for each nonzero $b \in \mathbb{C}$ there exists a structure (which we call harmonic) of an involutive algebraic 4-valued coset group $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$ with neutral element 0, whose addition is given by the polynomial

$$\begin{aligned} p_{4,\text{har}}(z; x, y) = & e_1^4 - 8 e_1^2 e_2 + 16 e_2^2 - 128 e_1 e_3 \\ & - 112 b e_1^2 e_3 - 4 b^2 e_1^3 e_3 - 64 b e_2 e_3 \\ & - 112 b^2 e_1 e_2 e_3 - 64 b^2 e_3^2 \\ & - 288 b^3 e_1 e_3^2 + 6 b^4 e_1^2 e_3^2 - 136 b^4 e_2 e_3^2 \\ & - 112 b^5 e_3^3 - 4 b^6 e_1 e_3^3 + b^8 e_3^4 \end{aligned}$$

where e_k denotes the k -th elementary symmetric function in x, y, z . All such 4-valued groups are isomorphic.

Proposition (Buchstaber, Kornev, 2025)

The iterating elements of the 4-valued group $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$ form the set $\{0, -1/b\}$, which is not any 4-valued subgroup (nor even a submonoid).

Proof. Let y be an iterating element in $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$. Then for any $x \in \mathbb{C}$ the discriminant of $p_{4,\text{har}}(z; x, y)$ vanishes:

$$\begin{aligned} & x^3 y^3 (bx + 1)^2 (by + 1)^2 (x - y)^2 \\ & \cdot (b^2 xy - 1)^2 (b^2 x^2 + 4b^2 xy + 2bx + 1)^2 \\ & \cdot (b^2 x^2 y + 2bxy + 4x + y)^2 (4b^2 xy + b^2 y^2 + 2by + 1)^2 \\ & \cdot (b^2 xy^2 + 2bxy + x + 4y)^2 = 0. \end{aligned}$$

Hence precisely 0 and $-1/b$ are iterating elements of $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$. Since $p_{4,\text{har}}(z; -1/b, -1/b) = 256 z^2/b^2$, the product $(-1/b) * (-1/b)$ is not defined in $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$ □.

6-Valued Structures on $\mathbb{CP}^1$

There is a unique projective equivalence class of nonsingular cubics (with j -invariant equal to 0) whose group of points is isomorphic to $\mathbb{Z}/6$.

For each complex number $c \neq 0$, the *equiharmonic cubic*

$$\mathcal{E} = \{y^2 = x^3 + c\}$$

belongs to this class [Dol12, Theorem 3.1.3].

Consider the map ($\varepsilon = e^{2\pi i/3}$)

$$\varphi_6 : \mathcal{E} \rightarrow \mathcal{E}$$

$$(x, y) \mapsto (\varepsilon x, -y),$$

$$\infty \mapsto \infty.$$

φ_6 is an automorphism of the abelian variety $\mathcal{E}$. The points of the orbit space $\mathcal{E} / \langle \varphi \rangle$ are identified with the fibers of the map:

$$\pi_6 : \mathcal{E} \rightarrow \mathbb{CP}^1$$

$$(x, y) \mapsto y^2,$$

$$\infty \mapsto \infty.$$

$$y_1 * y_2 = [\pi_6((x_1, y_1), \varphi_6^r(x_2, y_2)) \mid r = 0, \dots, 5]$$

$$= \left[\left(m_{\ell,k} \left(2\sqrt[3]{y_1 - c} + \varepsilon^k \sqrt[3]{y_2 - c} - m_{\ell,k}^2 \right) - \sqrt{y_1} \right)^2 \mid \begin{array}{l} k = 0, 1, 2; \\ \ell = 0, 1 \end{array} \right],$$

$$\text{where } m_{\ell,k} = \frac{\sqrt{y_1} - (-1)^\ell \sqrt{y_2}}{\sqrt[3]{y_1 - c} - \varepsilon^k \cdot \sqrt[3]{y_2 - c}}.$$

Theorem (Buchstaber, Kornev, 2025)

For each nonzero $c \in \mathbb{C}$, there exists a unique structure (*equiharmonic*) of an involutive 6-valued algebraic coset group $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$ with neutral element 0 and addition given by

$$\begin{aligned} p_{6,\text{eqh}}(z; -x, -y) = & e_1^6 - 2^2 \cdot 3 e_1^4 e_2 + 2^4 \cdot 3 e_1^2 e_2^2 - 2^6 e_2^3 - 2 \cdot 3^4 \cdot 17 e_1^3 e_3 \\ & - 2^3 \cdot 3^4 \cdot 19 e_1 e_2 e_3 + 3^3 \cdot 19^3 e_3^2 \\ & - 2^5 \cdot 3^2 \cdot 11 c e_1^4 e_3 - 2^2 \cdot 3^3 \cdot 5 c^2 e_1^5 e_3 - 2^4 \cdot 3^2 \cdot 211 c e_1^2 e_2 e_3 \\ & - 2^2 \cdot 3^3 \cdot 197 c^2 e_1^3 e_2 e_3 - 2^3 \cdot 3^5 c^3 e_1^4 e_2 e_3 - 2^6 \cdot 3^2 \cdot 5^2 c e_2^2 e_3 \\ & - 2^4 \cdot 3^3 \cdot 107 c^2 e_1 e_2^2 e_3 - 2^4 \cdot 3^7 c^3 e_1^2 e_2^2 e_3 - 2 \cdot 3^6 c^4 e_1^3 e_2^2 e_3 \\ & - 2^6 \cdot 3^5 c^3 e_2^3 e_3 - 2^3 \cdot 3^7 c^4 e_1 e_2^3 e_3 + 2^4 \cdot 3^3 \cdot 7^2 \cdot 19 c e_1 e_3^2 \\ & + 2^2 \cdot 3^3 \cdot 47 \cdot 53 c^2 e_1^2 e_3^2 + 2^3 \cdot 3^5 \cdot 61 c^3 e_1^3 e_3^2 + 2 \cdot 3^7 \cdot 17 c^4 e_1^4 e_3^2 \\ & + 2^2 \cdot 3^4 \cdot 701 c^2 e_2 e_3^2 + 2^3 \cdot 3^6 \cdot 7 \cdot 13 c^3 e_1 e_2 e_3^2 + 2^{10} \cdot 3^6 c^4 e_1^2 e_2 e_3^2 \\ & + 2^4 \cdot 3^8 \cdot 5 c^5 e_1^3 e_2 e_3^2 + 2 \cdot 3^7 \cdot 5 \cdot 17 c^4 e_2^2 e_3^2 + 2^5 \cdot 3^8 \cdot 7 c^5 e_1 e_2^2 e_3^2 \\ & + 2^2 \cdot 3^9 \cdot 17 c^6 e_1^2 e_2^2 e_3^2 + 2^2 \cdot 3^9 \cdot 11 c^6 e_2^3 e_3^2 + 2^3 \cdot 3^{11} c^7 e_1 e_2^3 e_3^2 \\ & + 3^{12} c^8 e_2^4 e_3^2 - 2^3 \cdot 3^{12} c^3 e_3^3 - 2 \cdot 3^9 \cdot 5 \cdot 47 c^4 e_1 e_3^3 \\ & - 2^4 \cdot 3^9 \cdot 17 c^5 e_1^2 e_3^3 - 2^2 \cdot 3^9 \cdot 5 c^6 e_1^3 e_3^3 - 2^6 \cdot 3^{11} c^5 e_2 e_3^3 \\ & - 2^2 \cdot 3^{10} \cdot 5 \cdot 11 c^6 e_1 e_2 e_3^3 - 2^3 \cdot 3^{11} c^7 e_1^2 e_2 e_3^3 - 2^3 \cdot 3^{11} \cdot 5 c^7 e_2^2 e_3^3 \\ & - 2 \cdot 3^{12} c^8 e_1 e_2^2 e_3^3 - 2^3 \cdot 3^{12} c^6 e_3^4 - 2^3 \cdot 3^{12} c^7 e_1 e_3^4 \\ & + 3^{12} c^8 e_1^2 e_3^4 - 2^2 \cdot 3^{13} c^8 e_2 e_3^4. \end{aligned}$$

Proposition (Buchstaber, Kornev, 2025)

The set $\{0, 1/c\}$ of iterating elements of the 6-valued group $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$ is not any 6-valued subgroup (nor even a submonoid).

Let y be an iterating element in $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$. Then for any $x \in \mathbb{C}$ the discriminant of the polynomial $p_{6,\text{eqh}}(z; x, y)$ vanishes:

$$x^5 y^5 (cx-1)^4 (cy-1)^4 (x-y)^4 (27c^2 x^3 + 81c^2 x^2 y - 54cx^2 - 18cxy + 27x + y)^4 \dots$$

Since

$$p_{6,\text{eqh}}(z; 1/c, 1/c) = 2^{18} z^3 (cz + 1)^3 / c^3,$$

we have:

$$1/c * 1/c = [0, 0, 0, -1/c, -1/c, -1/c].$$

Because the element $-1/c$ is not iterating, the set $0, 1/c$ is not any 6-valued submonoid of the group $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$.

Nodal Case

We now turn to singular cubics.

The change of variables

$$x \mapsto x + 1, \quad y \mapsto y + 1, \quad z \mapsto z + 1$$

shows that the Chebyshev $\mathbb{M}_{\text{mult}}(\mathbb{C}P^1)$ is isomorphic to the monoid $\mathbb{G}_{\mathbb{C}P^1}(B_a)$ with $a_1 = 1, a_2 = a_3 = 0$. This monoid corresponds to the cubic

$$\{y^2 = x^2(x + 1)\}.$$

Nodal Case

Let a be such that the polynomial

$$P(x) = x^3 + a_1x^2 + a_2x + a_3$$

has a double root. In this case the equation of the curve $\mathcal{E}$ takes the form ($\alpha \neq \beta$):

$$\mathcal{E} = \{y^2 = (x - \alpha)^2(x - \beta)\}.$$

Parametrize $\mathcal{E}$ by the slope m of the line passing through the node $\mathcal{O} = (\alpha, 0)$:

$$\begin{cases} x = m^2 + \beta, \\ y = m(m^2 + \beta - \alpha). \end{cases}$$

As before, there is a branched double covering

$$\mathcal{E} \rightarrow \mathbb{CP}^1 \quad (x(m), y(m)) \mapsto x(m), \quad \infty \mapsto \infty.$$

with branch points at α , β , and ∞ .

Nodal Case

Lemma

Let $m_1 \oplus m_2 = -m_3$ for points $m_1, m_2, m_3 \in \mathbb{CP}^1$ on the curve $\mathcal{E}$ with respect to the above parametrization. Then

$$m_3 = \frac{m_1 m_2 - m_+ m_-}{m_1 + m_2},$$

where $m_{\pm} = \pm\sqrt{\alpha - \beta}$ and $[m_1, m_2] \neq [m_+, m_-]$.

Proposition

The Möbius transformation $m \mapsto \frac{m+m_-}{m+m_+}$ establishes an isomorphism of 1-valued algebraic monoids $E \cong M_{\text{mult}}(\mathbb{CP}^1)$.

Nodal Case

Consider the involution ι on the monoid $\mathcal{E}(\mathbb{C})$ such that $\iota : m \mapsto -m$ for $m \in \mathbb{C}$ and $\iota(\infty) = \infty$. Then the orbit space $\mathbb{C}P^1 / \langle \iota \rangle$ is identified with $\mathbb{C}P^1$ via the map

$$\begin{aligned}\psi : \mathbb{C}P^1 &\rightarrow \mathbb{C}P^1 \\ z &\mapsto z^2.\end{aligned}$$

This gives a coset 2-valued algebraic monoid $\mathbb{M}_{\text{node}}(\mathbb{C}P^1) = \mathcal{E}_{\langle \iota \rangle}$

$$m_1 * m_2 = \left[\left(\frac{\sqrt{m_1} \sqrt{m_2} \pm a}{\sqrt{m_1} \pm \sqrt{m_2}} \right)^2 \right]$$

defined on the set $\text{Sym}^2(\mathbb{C}P^1) \setminus [a, a]$, where $a = \alpha - \beta$.

The values $m_1 * m_2$ are the roots of the quadratic trinomial

$$\begin{aligned}(m_1 - m_2)^2 z^2 - 2 \left(a^2(m_1 + m_2) - 4am_1m_2 + m_1m_2(m_1 + m_2) \right) z \\ + (a^2 - m_1m_2)^2.\end{aligned}$$

Nodal Case

Writing the addition law (50) in the original coordinates (x, y) of the curve $\mathcal{E}$, we obtain the algebraic law

$$x_1 * x_2 = \left[\left(\frac{\sqrt{x_1 - \beta} \sqrt{x_2 - \beta} \pm (\alpha - \beta)}{\sqrt{x_1 - \beta} \pm \sqrt{x_2 - \beta}} \right)^2 + \beta \right].$$

The values $x_1 * x_2$ are the roots (in z) of the Kontsevich polynomial $D_{\alpha, \beta}(-z; -x_1, -x_2) = (x_1 x_2 z)^2 B_{\alpha, \beta}(1/z; 1/x_1, 1/x_2)$, where

$$\begin{aligned} B_{\alpha, \beta}(z; x_1, x_2) = & \left((\alpha^2 x_1 x_2 - 1)^2 - 4\alpha\beta x_1 x_2 (\alpha x_1 - 1)(\alpha x_2 - 1) \right) z^2 \\ & - 2(-2\beta x_1 x_2 (\alpha x_1 - 1)(\alpha x_2 - 1) + \alpha x_1 x_2 (\alpha(x_1 + x_2) - 4) + x_1 + x_2) z \\ & + (x_1 - x_2)^2. \end{aligned}$$

Nodal Case

Theorem (Buchstaber, Kornev, 2025)

Let $\alpha \neq \beta \in \mathbb{C}$, and let $\mathcal{E} = \{y^2 = (x - \alpha)^2(x - \beta)\}$. Then the addition of points on $\mathcal{E}$ and the involution σ define on $\mathbb{CP}^1$ a unique structure of a 2-valued coset algebraic monoid $\mathbb{M}_{\text{node}}(\mathbb{CP}^1)$ with neutral element ∞ and operation

$$x_1 * x_2 = \left[\left(\frac{\sqrt{x_1 - \beta} \sqrt{x_2 - \beta} \pm (\alpha - \beta)}{\sqrt{x_1 - \beta} \pm \sqrt{x_2 - \beta}} \right)^2 + \beta \right],$$

given by the Kontsevich polynomial $D_{\alpha, \beta}(-z; -x_1, -x_2)$. The element α is absorbing, i.e., $x * \alpha = \alpha * x = \alpha$ for any $x \in \mathbb{CP}^1 \setminus \{\alpha\}$. The product $\alpha * \alpha$ is not defined. Moreover, the element 0 has an inverse ($\text{inv}(0) = 0, 0 * 0 = [\alpha(1 - \alpha/(4\beta)), \infty]$) if and only if $\alpha \neq 0$.

Doubling Points

Proposition (Buchstaber, Kornev, 2025)

The set of doubling points for $\mathbb{M}_{\text{node}}(\mathbb{C}P^1)$ consists of three points: ∞ , α , and β .

Cuspidal Case

Finally, consider the case of a triple root

$$\mathcal{E} = \{y^2 = (x - \alpha)^3\}.$$

The Lemma (in the limit $\beta \rightarrow \alpha$) immediately yields:

Lemma

The addition law on the elliptic curve $\mathcal{E} = \{y^2 = (x - \alpha)^3\}$ has the form:

$$m_3 = \frac{m_1 m_2}{m_1 + m_2}, \text{ where } m = y/(x - \alpha).$$

Proposition

The Möbius transformation $m \mapsto 1/m$ establishes an isomorphism between the 1-valued algebraic monoids $E(\mathbb{C}) \cong M_{\text{cusp}}(\mathbb{CP}^1)$.

Cuspidal Case

The set $\mathbb{M}_{\text{node}}(\mathbb{CP}^1)$ tends, as $\alpha \rightarrow \beta$, to the monoid $\mathbb{M}_{\text{cusp}}(\mathbb{CP}^1)$ with identity ∞ and multiplication

$$x_1 * x_2 = \left[\frac{(x_1 - \alpha)(x_2 - \alpha)}{(\sqrt{x_1 - \alpha} \pm \sqrt{x_2 - \alpha})^2} + \alpha \right].$$

Proposition

The monoids $\mathbb{M}_{\text{cusp}}(\mathbb{CP}^1)$ and $\mathbb{M}_2(\mathbb{CP}^1)$ are isomorphic.

Proof. The map

$$x \mapsto \frac{1}{x - \alpha}$$

defines an isomorphism $\mathbb{M}_{\text{cusp}}(\mathbb{CP}^1) \rightarrow \mathbb{M}_2(\mathbb{CP}^1)$. □

Theorem (Buchstaber, Kornev, 2025)

Let $\alpha \in \mathbb{C}$ and let $\mathcal{E} = \{y^2 = (x - \alpha)^3\}$.

(I) The addition of points on $\mathcal{E}$ and the involution define on $\mathbb{C}P^1$ a unique structure of a 2-valued coset algebraic monoid $\mathbb{M}_2(\mathbb{C}P^1)$.

(II) The group $\mathbb{G}_{3,\text{eqh}}(\mathbb{C}P^1)$ tends, as $c \rightarrow 0$, to the monoid $\mathbb{M}_3(\mathbb{C}P^1)$.

(III) The group $\mathbb{G}_{4,\text{har}}(\mathbb{C}P^1)$ tends, as $b \rightarrow 0$, to the monoid $\mathbb{M}_4(\mathbb{C}P^1)$.

(IV) The group $\mathbb{G}_{6,\text{eqh}}(\mathbb{C}P^1)$ tends, as $c \rightarrow 0$, to the monoid $\mathbb{M}_6(\mathbb{C}P^1)$.

Proposition (Buchstaber, Kornev, 2025)

The set of doubling points for $\mathbb{M}_{\text{cusp}}(\mathbb{C}P^1)$ consists of two points: ∞ and α .

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Thank you for attention!