

n -Valued Groups, Kronecker Sums, and Wendt Matrices*

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A Brief History

In 1971, V. M. Buchstaber and S. P. Novikov proposed a construction *motivated by the theory of characteristic classes* [BN71].

This construction describes a multiplication in which *the product of any pair of elements is a multiset of n points* [Buc06].

An axiomatic definition of n -valued groups and the first results of their algebraic theory were obtained in a subsequent series of works by V. M. Buchstaber.

Currently, the theory of n -valued (formal, finite, discrete, topological, and algebro-geometric) groups and their applications in various areas of mathematics and mathematical physics are being developed by a number of authors.

n -Valued Monoids

An *n -valued monoid* is a set X equipped with an operation

$$* : X \times X \rightarrow \text{Sym}^n(X)$$

where $\text{Sym}^n(X) = X^{\times n}/S_n$ is the set of unordered n -tuples of elements of X :

- *Associativity.* The n^2 -multisets

$$\begin{aligned} [x * w \mid w \in y * z], \\ [w * z \mid w \in x * y] \end{aligned}$$

coincide.

- *Unit.* There exists an element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for every $x \in X$.

n -Valued Groups

An *n -valued group* X is an n -valued monoid equipped with an inverse map

$$\text{inv} : X \rightarrow X$$

that is, a map such that for every $x \in X$

$$x * \text{inv}(x) \ni e, \quad \text{inv}(x) * x \ni e.$$

The notions of homomorphisms, commutativity, kernels, and related concepts admit natural generalizations from the context of 1-valued groups to that of n -valued groups.

Groups $\mathbb{X}_n$

For each $n \geq 3$, there exists a commutative n -valued group $\mathbb{X}_n$ on the set $\{0, 1, 2\}$ with unit 0 [Kor25, Example 3]:

$$1 * 1 = [0, \underbrace{2, \dots, 2}_{n-1}]$$

$$1 * 2 = [0, 1, \underbrace{2, \dots, 2}_{n-2}]$$

$$2 * 2 = [0, 1, 1, \underbrace{2, \dots, 2}_{n-3}]$$

The inverse map inv may be defined in four different ways:
 $\text{Inv}(1) = \text{Inv}(2) = \{1, 2\}$.

Coset (Étale) Construction

Given 1-valued group G with multiplication μ_0 , identity e_G , and inverse $\text{inv}_G(u) = u^{-1}$, and given $H < \text{Aut}(G)$, $|H| = n$.

Consider *the set of orbits* $X := G/H$ with projection $\pi : G \rightarrow X$ and an n -valued multiplication

$$\mu : X \times X \rightarrow \text{Sym}^n(X)$$

$$\mu(x, y) = [\pi(\mu_0(u, h(v))) \mid h \in H]$$

where $u \in \pi^{-1}(x)$ and $v \in \pi^{-1}(y)$.

Theorem (Buchstaber, [Buc06])

The multiplication μ defines an n -valued multiplication on X with identity $e_x = \pi(e_G)$ and inverse $\text{inv}_X(x) = \pi(\text{inv}_G(u))$, where $u \in \pi^{-1}(x)$.

Groups $\mathbb{G}_n$

Consider the operation $* : \mathbb{C} \times \mathbb{C} \rightarrow \text{Sym}^n(\mathbb{C})$ that assigns to each pair of complex numbers x and y the multiset

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n \mid r = 1 \dots n]$$

of n complex numbers, where ε is a primitive n th root of unity, and $\sqrt[n]{\cdot}$ denotes a fixed root branch.

The operation $*$, the identity element 0 , and the inverse $\text{inv}(x) = (-1)^n x$ endow the set of complex numbers $\mathbb{C}$ with the structure of a *commutative n -valued group* $\mathbb{G}_n$.

Groups $\mathbb{G}_n$

$\mathbb{G}_n$'s are examples of coset groups.

Let G be the additive subgroup of $\mathbb{C}$. For a fixed a primitive n th root of unity ε , consider an automorphism

$$\varphi : z \mapsto \varepsilon z$$

We have the projection

$$\begin{aligned}\pi : \mathbb{C} &\rightarrow \mathbb{C} / \langle \varphi \rangle \cong \mathbb{C} \\ z &\mapsto z^n\end{aligned}$$

So, in this case we get

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n \mid r = 1 \dots n]$$

§ 2b. Обратим теперь внимание на то обстоятельство, что степенная система $B^{-1}(k^2 B(x)) = k^2 \Psi^k(u\bar{u})$ связана со своеобразной «двузначной формальной группой»

$$F^{\pm}(x, y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2),$$

в которой операция возведения в степень однозначна, а именно, $B^{-1}(k^2 B(x)) = F^{\pm}(x, \underbrace{\dots, x}_{k \text{ штук}})$. Если $x = u\bar{u}$, $y = v\bar{v}$, то $F^{\pm}(x, y) = \{|f(u, v)|^2; |f(u, \bar{v})|^2\}$ и для $U(1)$ -расслоений ξ, η над $CF^{\infty} \times CP^{\infty}$, где $u = \sigma_1(\xi)$, $v = \sigma_1(\eta)$, мы имеем $F^{\pm}(x, y) = \{\sigma_2(\xi\eta + \overline{\xi\eta}); \sigma_2(\xi\bar{\eta} + \overline{\xi\eta})\}$, $x = \sigma_2(\xi + \bar{\xi})$, $y = \sigma_2(\eta + \bar{\eta})$.

Excerpt from [BN71]

Non-coset Groups

For every n -valued group on a set X , in addition to the inverse map $\text{inv} : X \rightarrow X$, we define a multivalued inverse map $\text{Inv} : X \rightarrow \mathcal{P}(X)$ taking values in the set of all finite subsets of X :

$$\text{Inv}(x) = \{y \in X \mid x * y \ni e, \quad y * x \ni e\}.$$

An n -valued group has a uniquely defined inverse map if and only if $\text{inv} = \text{Inv}$. We call such groups *reversible*; otherwise, we call them *non-reversible*.

Every coset group is reversible. So, $\mathbb{X}_n$ is not a coset group.

Combinatorial Algebras

An n -valued group $(X, *, \text{inv})$ is called *★-involutive* if the inverse map $x \mapsto \text{inv}(x)$ is an involution satisfying the following conditions:

- 1 For any $x \in X$, one has $x * y \ni e$ if and only if $y = \text{inv}(x)$.
- 2 $m(x) = m(\text{inv}(x))$, where $m(x) = m_{x, \text{inv}(x)}^e$ is the multiplicity of e in the product $x * \text{inv}(x)$.
- 3 $\text{inv}(x * y) = \text{inv}(y) * \text{inv}(x)$ for all $x, y \in X$.

The category of combinatorial algebras and the category of multivalued involutive groups are equivalent [BVEP96].

The Classification of n -Valued Monoids

The complete classification of all $\star$ -involutive n -valued groups of order 3 was achieved in [Pon24, GPV25], based on the classification of all single-valued finite groups of rank 3.

A full classification of n -valued monoids and groups of order 3 has been recently obtained by Kornev in [arXiv:2508.04454](https://arxiv.org/abs/2508.04454) [Kor25].

As a corollary, *there are non- $\star$ -involutive non-coset n -valued groups*.

Back to the Group $\mathbb{G}_n$

Recall the operation $*$: $\mathbb{C} \times \mathbb{C} \rightarrow \text{Sym}^n(\mathbb{C})$

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n \mid r = 1 \dots n]$$

where ε is a primitive n th root of unity, and $\sqrt[n]{\cdot}$ denotes a fixed root branch.

The operation $*$, the identity element 0, and the inverse $\text{inv}(x) = (-1)^n x$ endow the set of complex numbers $\mathbb{C}$ with the structure of a *commutative n -valued group* $\mathbb{G}_n$.

Introducing Polynomials $p_n(z; x, y)$

Introduce the following symmetric polynomials:

$$p_n(z; x, y) = \prod_{w \in \text{inv}(x) * \text{inv}(y)} (z - w)$$

$$p_1 = \sigma_1,$$

$$p_2 = \sigma_1^2 - 4\sigma_2,$$

$$p_3 = \sigma_1^3 - 27\sigma_3,$$

$$p_4 = \sigma_1^4 - 2^3 \sigma_1^2 \sigma_2 + 2^4 \sigma_2^2 - 2^7 \sigma_1 \sigma_3,$$

$$p_5 = \sigma_1^5 - 5^4 \sigma_1^2 \sigma_3 - 5^5 \sigma_2 \sigma_3,$$

$$p_6 = \sigma_1^6 - 2^2 \cdot 3 \sigma_1^4 \sigma_2 - 2 \cdot 3^4 \cdot 17 \sigma_1^3 \sigma_3 + 2^4 \cdot 3 \sigma_1^2 \sigma_2^2 \\ - 2^3 \cdot 3^4 \cdot 19 \sigma_1 \sigma_2 \sigma_3 - 2^6 \sigma_2^3 + 3^3 \cdot 19^3 \sigma_3^2$$

where σ_j 's are elementary symmetric polynomials in x, y, z .

Introducing Polynomials $p_n(z; x, y)$

For $p_{18}(z; x, y)$ we have:

$$(18, 0, 0) \rightarrow 1$$

$$(16, 1, 0) \rightarrow -2^2 3^2$$

$$(15, 0, 1) \rightarrow -2^1 3^5 43^1 293^1 13339^1$$

$$(14, 2, 0) \rightarrow 2^6 3^2$$

$$(13, 1, 1) \rightarrow -2^1 3^5 39079^1 30478663^1$$

$$(12, 3, 0) \rightarrow -2^8 3^1 7^1$$

$$(12, 0, 2) \rightarrow 3^4 167^1 58369^1 1702940402507^1$$

$$(11, 2, 1) \rightarrow -2^2 3^6 57769225879741^1$$

$$(10, 4, 0) \rightarrow 2^9 3^2 7^1$$

$$(10, 1, 2) \rightarrow -2^2 3^5 7^1 97^1 36913^1 180317^1 3375001057^1$$

$$(9, 3, 1) \rightarrow -2^1 3^5 23^1 144589^1 5245247209^1$$

$$(9, 0, 3) \rightarrow -2^1 3^8 5^1 12713^1 76919^1 37764598382689403^1$$

$$(8, 5, 0) \rightarrow -2^{11} 3^2 7^1$$

$$(8, 2, 2) \rightarrow 3^5 67589^1 626540941303495210351^1$$

etc.

where, e. g., $(16, 1, 0) \rightarrow -2^2 3^2$ means that the coefficient of $x^{16}y$ equals $-2^2 3^2$.

Introducing Polynomials $p_n(z; x, y)$

One can define the n -valued group $\mathbb{G}_n$ via the roots of polynomials:

$$\tilde{p}_n = p_n(z; (-1)^n x, (-1)^n y)$$

$$x * y = [z \mid z \text{ is a root of } \tilde{p}_n(z; x, y)]$$

These polynomials were first introduced in [BN71] in the context of the theory of n -valued groups.

The problem of describing the structure of the coefficients of the polynomials p_n was posed in [Buc06], where also the explicit form of $p_n(z; x, y)$ was given for $1 \leq n \leq 7$.

Kronecker Sums

Recall the following definition:

Let A and B be arbitrary square matrices over the field $\mathbb{k}$ of orders m and n , respectively.

The *Kronecker sum* $A \boxplus B$ of the matrices $A \in \text{Mat}_{m \times m}(\mathbb{k})$ and $B \in \text{Mat}_{n \times n}(\mathbb{k})$ is the $mn \times mn$ matrix

$$A \boxplus B = A \otimes I_n + I_m \otimes B.$$

This notion has many applications in various fundamental and applied fields of science, including differential equations, quantum computing, machine learning, robotics, and others.

Kronecker Sums

Proposition (Stephanos, [Sté00])

Let $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$, and $p(x, y) = \sum_{j,k=0}^{\ell} c_{jk} x^j y^k \in \mathbb{C}[x, y]$.

Consider the matrix

$$p(A, B) = \sum_{j,k=0}^{\ell} c_{jk} A^j \otimes B^k$$

Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of the matrix A ,
 $\mu_1, \dots, \mu_n$ the eigenvalues of B .

Then the eigenvalues of the matrix $p(A, B)$ are the mn numbers
 $p(\lambda_r, \mu_s)$, where $r = 1, \dots, m$ and $s = 1, \dots, n$.

Frobenius Companion Matrices

Let $f(t)$ be $t^m + a_{m-1}t^{m-1} + \cdots + a_0 \in \mathbb{C}[t]$.

The *Frobenius companion matrix* $F(a_{m-1}, \dots, a_0)$ of the polynomial $f(t)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & 0 & -a_2 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{m-1} \end{pmatrix}.$$

$p_n(z; x, y)$ and Kronecker Sums

Proposition (Buchstaber, Kornev, 2025)

The following statements hold for each natural n :

❶ $p_n(z; x, y)$ is symmetric in the variables x , y , and z .

❷

$$p_n(z^n; x, y) = \chi(\underbrace{F(0, \dots, 0, (-1)^{n+1}x)}_n \boxplus \underbrace{F(0, \dots, 0, (-1)^{n+1}y)}_n; z)$$

❸ *The polynomial $p_n(z; x, y)$ is a homogeneous polynomial in the variables x , y , and z of degree n with integer coefficients.*

$p_n(z; x_1, \dots, x_m)$ and Kronecker Sums

Theorem (Buchstaber, Kornev, 2025)

The following statements hold for each natural n :

- ❶ $p_n(z; x_1, \dots, x_m)$ is symmetric in all variables.
- ❷ For all complex $z, x_1, \dots, x_m$, we have the equality

$$p_n(z^n; x_1, \dots, x_m) = \chi(F(\underbrace{0, \dots, 0}_{n}, (-1)^{n+1}x_1) \boxplus \dots \boxplus F(\underbrace{0, \dots, 0}_{n}, (-1)^{n+1}x_m); z).$$

- ❸ $p_n(z; x_1, \dots, x_m)$ is homogeneous in the variables $z, x_1, \dots, x_m$ of degree n^{m-1} with integer coefficients.

θ -Circulants

For any elements $a_0, \dots, a_{n-1} \in \mathbb{k}$, introduce a θ -circulant matrix:

$$\text{Circ}_{\theta}(a_0, \dots, a_{n-1}) = \begin{pmatrix} a_0 & \theta a_1 & \theta a_2 & \cdots & \theta a_{n-1} \\ a_{n-1} & a_0 & \theta a_1 & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \theta a_2 \\ \vdots & \ddots & \ddots & \ddots & \theta a_1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

Wendt Matrices

This class contains *Wendt matrices* W_n of the form

$$\begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ \binom{n}{n-1} & 1 & \binom{n}{n-2} & \cdots & \binom{n}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & 1 \end{pmatrix}$$

Proposition (Wendt, [Wen94])

Let p be an odd prime and $q = 2kp + 1$ be prime for some $k \geq 1$. Then there exist integers x, y, z , not divisible by q , such that

$$x^p + y^p + z^p \equiv 0 \pmod{q}$$

if and only if q divides $\det W_{2k}$.

Wendt Matrices

Following tradition, we say that the *first case of Fermat's theorem is true for the odd prime exponent p* when:

if x, y, z are (non-zero) integers, not multiples of p , then $x^p + y^p \neq z^p$.

Theorem (Germain, Legendre, 1823, [Rib99])

Let p and q be distinct odd prime numbers, and suppose the following conditions hold:

- *If a, b, c are integers and $a^p + b^p + c^p \equiv 0 \pmod{q}$, then $q \mid abc$.*
- *There is no integer m such that $m^p \equiv p \pmod{q}$.*

Then the first case of Fermat's Last Theorem holds for the exponent p .

$p_n(z; x, y)$ and Wendt Matrices

Theorem (Buchstaber, Kornev, 2025)

The polynomial $p_n(z; x, y)$ defining the n -valued multiplication is the determinant of a y -circulant $n \times n$ matrix:

$$\text{Circ}_y \left(w^n + (-1)^{n+1}x + y, \binom{n}{1}w, \dots, \binom{n}{n-1}w^{n-1} \right)$$

where $w^n = z$.

These matrices generalize the Wendt matrices (the substitution $x = (-1)^n$, $y = z = 1$).

$p_n(z; x, y)$ and Wendt Matrices

$$p_2 = \begin{vmatrix} w^2 - x + y & 2wy \\ 2w & w^2 - x + y \end{vmatrix}$$

$$p_3 = \begin{vmatrix} w^3 + x + y & 3yw & 3yw^2 \\ 3w^2 & w^3 + x + y & 3yw \\ 3w & 3w^2 & w^3 + x + y \end{vmatrix}$$

$$p_4 = \begin{vmatrix} w^4 - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & w^4 - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & w^4 - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & w^4 - x + y \end{vmatrix}$$

The Coefficients of $p_n(z; x, y)$

Theorem (Buchstaber, Kornev, 2025)

For prime $n \geq 5$, the polynomial

$$p_n(z; x, y) - (x + y + z)^n$$

is divisible by n^4xyz .

This result follows from the above observations and from the following:

Theorem (Wolstenholme, [Wol62])

$$\binom{2n-1}{n-1} - 1$$

is divisible by n^3 for primes $n \geq 5$.

The Irreducibility of $p_n(z; x, y)$

Theorem (Buchstaber, Kornev, 2025)

The polynomials $p_n(z; x_1 \dots x_m)$ in the variables $z_1 \dots x_1 \dots x_m$ are irreducible over $\mathbb{C}$ for each positive integer n .

Theorem (Buchstaber, Kornev, 2025)

Fix a positive integer $n \geq 2$. Let the positive integer pairwise coprime numbers $x_1 \dots x_m$ be such that $(\sqrt[n]{x_j})^k$ is irrational for all positive integer $k < n$ for each $j = 1 \dots m$. Then the polynomial $p(z; x_1 \dots x_m)$ in the variable z is irreducible over $\mathbb{Q}$.

p_n and Discriminants

Theorem ([GRS24], direct proof in [BK25])

The discriminant $\text{disc}(P_{x,y,z})$ of the polynomial

$$P_{x,y,z}(T) = ((-1)^n x T^{n-1} + (-1)^n y)(1 + T)^{n-1} - T^{n-1} z$$

with respect to the variable T , which is a polynomial of degree $4n - 6$, is related to the polynomial $p_n(z; x, y)$ as follows:

$$(-1)^n (n-1)^{2n-2} (xyz)^{n-2} p_n(z; x, y) = \text{disc}_T(P_{x,y,z}(T))$$

for each $n \geq 2$.

$\text{disc}(P_{x,y,z})$ is the discriminant of the homogenized polynomial

$$(u + v)^{n-1} (xu^{n-1} + yv^{n-1}) - (uv)^{n-1} z$$

with respect to the variables u and v .

p_n and Discriminants

We get the following:

Corollary (Helou, [Hel97])

For each odd $n \geq 3$, the discriminant $\text{disc } Q(t)$ of the polynomial $Q(t) = (1 + t)^n - t^n - 1$ and the determinant $\det W_{n-1}$ of the Wendt matrix are related by the following equality:

$$\text{disc } Q(t) = (-1)^{\frac{n-1}{2}} n^{n-2} \det W_{n-1}.$$

p_n and Discriminants

Consider the homogeneous polynomial

$$P_x(u) = (u_1 \dots u_m)^{n-1} \left(z + (-1)^n \left(\sum_{j=1}^m u_j \right)^{n-1} \cdot \sum_{j=1}^m \frac{x_j}{u_j^{n-1}} \right)$$

of degree $m(n-1)$ in the variables $u_1 \dots u_m$.

The discriminant $\text{disc}(P_x(u))$ of the polynomial $P_x(u)$ is a homogeneous polynomial of degree $m(m(n-1)-1)^{m-1}$.

p_n and Discriminants

From [GRS24, Theorem 3.2], one gets

Theorem ([GRS24])

The discriminant $\text{disc } P_x(u)$ of the homogeneous polynomial $P_x(u)$, which has degree $m(n-1)$ in the variables $u_1 \dots u_m$, is divisible by certain powers of the monomial $zx_1 \dots x_m$ and the polynomial $p_n(z; x_1 \dots x_m)$.

n -Valued Groups and Representations

Let a_{jk}^ℓ be the multiplicities in the decomposition of the tensor product $\rho_j \otimes \rho_k$ into a sum of irreducible finite dimensional complex representations $\{\rho_\ell\}$ of a finite group G :

$$\rho_j \otimes \rho_k = \sum_{\ell=0}^k a_{jk}^\ell \rho_\ell.$$

Theorem (Buchstaber, [Buc06])

Let $d_j = \dim \rho_j$, $n = \text{lcm}(d_j d_k \mid j, k = 0 \dots k)$, and $m_{jk}^\ell = n a_{jk}^\ell d_\ell / (d_j d_k)$. Then on the set $\left\{ \frac{\rho_j}{\dim \rho_j} \right\}$ there is a commutative n -valued group structure $\mathbb{G}_\otimes^{\text{rep}}$, such that the element $\rho_\ell / \dim \rho_\ell$ appears in the n -set $\rho_j / \dim \rho_j * \rho_k / \dim \rho_k$ with multiplicity m_{jk}^ℓ .

n -Valued Groups and Representations

For the Kronecker sum, we have:

$$\frac{\rho_j}{d_j} + \frac{\rho_k}{d_k} = \frac{\rho_j \boxplus \rho_k}{d_k d_j}.$$

Theorem (Buchstaber, Kornev, 2025)

Let G be a finite group. The set $\mathbb{G}_{\boxplus}^{\text{rep}}$, equipped with the above operation, neutral element 0 , and formal inverse $\text{inv}(\psi / \dim \psi) = -\psi / \dim \psi$, is a 1-group isomorphic to $\mathbb{Z}^N$, where N is the number of conjugacy classes of the group G .

Algebraic n -Valued Groups

Consider a polynomial over $\mathbb{C}$:

$$P(z; x, y) = \sum_{k=0}^n F_{n-k}(x, y) z^k$$

where $F_{n-k}(x, y) \in \mathbb{C}[x, y]$ and

$$F_0(0, 0) = 1 \tag{1}$$

Introduce an operation $*$, such that for $x, y \in \mathbb{C}$ with $F_0(x, y) \neq 0$:

$$x * y = [z \mid P(z; x, y) = 0]$$

In this context, let us formulate the axioms from the definition of n -valued groups.

Algebraic n -Valued Groups

From the *neutral element axiom* with $x = 0$, it follows that

$$P(z; 0, y) = F_0(0, y)(z - y)^n \quad (2)$$

The *inverse element axiom*:

$$F_n(x, \text{inv}(x)) = F_n(\text{inv}(x), x) = 0 \quad (3)$$

Denote by $z_j = z_j(x, y)$ the roots of P . Then the *associativity condition*:

$$\prod_{j,k=1}^n (t - z_j(z_k(x, y), z)) = \prod_{j,k=1}^n (t - z_j(x, z_k(y, z))) \quad (4)$$

Note that the coefficients of the polynomials in the formal variable t on both sides of equation (4) are polynomials in x , y , and z .

Algebraic n -Valued Groups

An *n -valued algebraic group* is an algebraic variety X with an n -valued multiplication defined on a dense subset $Y \subset X \times X$ in the Zariski topology.

An *n -algebraic n -valued group* on $\mathbb{C}$ is the set $\mathbb{C}$ with

$$\begin{aligned} * : (\mathbb{C} \times \mathbb{C}) \setminus \{(x, y) \mid F_0(x, y) = 0\} &\longrightarrow \text{Sym}^n(\mathbb{C}) \\ (x, y) &\longmapsto [\text{roots of } P(z; x, y) = 0] \end{aligned}$$

such that each of the variables x and y appears in it with degree at most n , and the polynomial satisfies conditions (1)–(4).

We denote the corresponding group by $\mathbb{G}(P)$.

n -Algebraic n -Valued Groups: Examples

- 1 The polynomial $P(z; x, y) = z - x - y$ defines a 1-algebraic 1-valued group on $\mathbb{C}$ with the usual addition as the operation.

- 2 Consider

$$P(z; x, y) = -x - y + z + \alpha xyz,$$

where $\alpha \in \mathbb{C}$. Then we have a 1-algebraic 1-valued commutative group on $\mathbb{C}$ with the operation

$$x * y = \frac{x + y}{1 + \alpha xy}.$$

$$\begin{pmatrix} 1 & x \\ \alpha x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ \alpha y & 1 \end{pmatrix} = (1 + \alpha xy) \begin{pmatrix} 1 & \frac{x+y}{1+\alpha xy} \\ \alpha \frac{x+y}{1+\alpha xy} & 1 \end{pmatrix}$$

n -Algebraic n -Valued Groups: Examples

Continuing with the example $P(z; x, y) = -x - y + z + \alpha xyz$.

For $\alpha > 0$, the exponent of the resulting formal group is

$$\exp_P(x) = \frac{\tanh(\alpha x)}{\sqrt{\alpha}}.$$

The corresponding Q -series is

$$Q_P(x) = \frac{\sqrt{\alpha} x}{\tanh(\sqrt{\alpha} x)}.$$

So, for $\alpha = 1$, the Q -series $Q_P(x)$ gives the *L -genus* or the *signature* of a stably complex $2n$ -manifold.

Symmetric n -algebraic n -valued Groups

An n -algebraic n -valued group is called *symmetric* if the polynomial

$$P(z; (-1)^n x, (-1)^n y)$$

is symmetric in the variables x , y , and z .

Let $P(z; x, y)$ define the multiplication in a symmetric n -algebraic n -valued group. Then the following hold:

- $F_0(0, y) \equiv 1$
- $F_n(x, y) = (x - y)^n$
- $\text{inv}(x) = (-1)^n x$

Symmetric n -algebraic n -valued Groups

$\mathbb{G}_n$ is $\mathbb{G}(p_n)$ corresponding to the integer polynomial

$$p_n(z; (-1)^n x, (-1)^n y) = \prod_{r=1}^n (z - (\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n),$$

where ε is a primitive n th root of unity.

Symmetric 2-algebraic 2-valued Groups

For $n = 2$, each polynomial P_2 can be written in the form

$$P_2(z; -x, -y) = \sigma_1^2 - 4\sigma_2 + k_2\sigma_3 + k_8\sigma_3^2 + k_4\sigma_1\sigma_3 + k_6\sigma_2\sigma_3$$

for indeterminate graded complex coefficients k_j with $\deg k_j = -j < 0$.

Theorem (Buchstaber, Veselov, [BV19]; Kornev, 2025)

The universal symmetric 2-algebraic 2-valued group $\mathbb{G}(P_2)$ over $\mathbb{C}$ is defined over the coefficient ring

$$\mathbb{C}[k_2, k_4, k_6, k_8]/(4k_8 - k_4^2 + k_6k_2) \cong \mathbb{C}[k_2, k_4, k_6],$$

i.e., the universal family of polynomials P_2 has the form:

$$P_2(z; x, y) = \sigma_1^2 - 4\sigma_2 + 2k_2\sigma_3 + (k_4^2 - k_6k_2)\sigma_3^2 + 2k_4\sigma_1\sigma_3 + 2k_6\sigma_2\sigma_3.$$

Symmetric 3-algebraic 3-valued Groups

The general form of the polynomials P_3 is as follows:

$$\begin{aligned} &\sigma_1^3 + k_0\sigma_3 + k_6\sigma_3^2 + k_{12}\sigma_3^3 + k_2\sigma_1\sigma_3 + k_4\sigma_2\sigma_3 + \\ &+ \ell_6\sigma_1\sigma_2\sigma_3 + \ell_4\sigma_1^2\sigma_3 + k_8\sigma_2^2\sigma_3 + \ell_8\sigma_3^2\sigma_1 + k_{10}\sigma_3^2\sigma_2, \end{aligned}$$

where the undetermined coefficients are graded by $\deg(k_i) = -i$, $\deg(\ell_j) = -j$, and each of the variables x , y , and z has degree 2.

Symmetric 3-algebraic 3-valued Groups

Theorem (Buchstaber, Kornev, 2025)

There are two classes over $\mathbb{C}$:

- ① *If $k_0 = -27$, then for $c \in \mathbb{C}$,*

$$P_3(z; -x, -y) = \sigma_1^3 - 27\sigma_3 + 18c\sigma_1^2\sigma_3 - 54c\sigma_2\sigma_3 \\ - 27c^2\sigma_2^2\sigma_3 + 81c^2\sigma_1\sigma_3^2.$$

Its homogeneous degree-three component is $P_3^h(z; -x, -y) = \sigma_1^3 - 27\sigma_3$; the full law gives $\mathbb{G}_3$ when $c = 0$.

- ② *If $k_0 = 0$, then for $\alpha \in \mathbb{C}$,*

$$P_3(z; -x, -y) = (\sigma_1 + \alpha\sigma_3)^3.$$

This is the diagonal law of the 1-algebraic 1-valued group G defined by $P_1(z; -x, -y) = \sigma_1 + \alpha\sigma_3$:

$$G \times G \xrightarrow{\mu} G \xrightarrow{\text{Sym}^3 \circ \text{diag}} \text{Sym}^3(G).$$

Thank you!

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