

Cartan Geometries

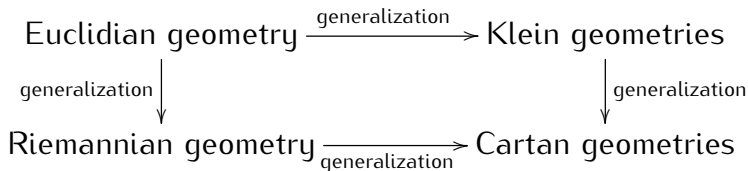
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A General Scheme



Achtung! This does not mean that any Cartan geometry locally looks like a Klein geometry. The obstruction is the curvature.

Rather, a Cartan geometry is constructed from a Klein geometry in a certain way.

The Fundamental Theorem of Calculus

Theorem (The local version)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\omega : M \longrightarrow \mathfrak{g}$ satisfying the structural equation

$$\Omega := d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Then, for each point $x \in M$ there exists a neighborhood U of x and a smooth map $f : U \longrightarrow G$ such that $\omega|_U = f^*\omega_G$.

In the theorem, the form ω is called *the Darboux derivative* of a map f .

The Fundamental Theorem of Calculus

The key ingredient of a global version is a development technique.

Definition (The simplest case of a segment)

It is a map $f : I \longrightarrow G$ satisfying $f(0) = g \in G$ and $f^*(\omega_G) = \omega$ is called the *development* of ω on G along I starting at $g \in G$.

It can be generalized to the case of general manifolds as follows:

Definition (The general case of manifolds)

Let ω be a form on M and $\gamma : (I, 0, 1) \longrightarrow (M, p, q)$ be a path.

A path $\tilde{\gamma} : (I, 0) \longrightarrow (G, p)$ is called the *development* of ω along γ starting at $g \in G$ if

$$\tilde{\gamma}^* \omega_G = \gamma^* \omega.$$

The properties of the development

One can show the development operation is correctly defined on the homotopy classes of paths for any $\omega \in \Omega^1(TM) \otimes \mathfrak{g}$ satisfying Maurer–Cartan equations.

The proof is derived from the local version of the Fundamental Theorem of Calculus (as expected).

Restricting to the case of loops, we get a homomorphism map called a *monodomy*

$$\Phi_\omega : \pi_1(M) \longrightarrow G.$$

The Fundamental Theorem of Calculus

Theorem (The global version)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\omega : M \longrightarrow \mathfrak{g}$ satisfying the structural equation

$$\Omega := d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Then, there is a bijection

$\{\omega \text{ is the Darboux derivative of some map } M \rightarrow G\}$



$\{\omega \mid d\omega + \frac{1}{2}[\omega, \omega] = 0 \text{ and } \Phi_\omega : \pi_1(M) \rightarrow G \text{ is trivial}\}$

This result has many fruitful applications.

Darboux Contexts

Definition

A Darboux context is a triple $(\mathfrak{g}, P, \omega_P)$ consisting of a Lie algebra $\mathfrak{g}$, a manifold P , and a $\mathfrak{g}$ -valued 1-form ω_P on P , s.t:

- ❶ *Regularity:* $\omega : T_p P \longrightarrow \mathfrak{g}$ is an isomorphism for all $p \in P$,
- ❷ *Structural equation:* $d\omega_P + \frac{1}{2}[\omega_P, \omega_P] = 0$,
- ❸ *Completeness:* any map $F : \mathbb{R} \longrightarrow \mathfrak{g}$ generates a complete vector field $(\omega_P^{-1}(F(t)), \partial_t)$ on $P \times \mathbb{R}$.

For example, each Lie group G canonically generates a Darboux context $(\mathfrak{g}, G, \omega_G)$.

Darboux Contexts

Definition

Let $(\mathfrak{g}, P, \omega_P)$ be a Darboux context and $f : M \longrightarrow P$ a smooth map.

A *Darboux derivative* of f is a $\mathfrak{g}$ -valued 1-form $f^*\omega_P$ on M .

The Fundamental Theorem Revisited

Theorem

Let $(P, \mathfrak{g}, \omega_P)$ be a Darboux context with P connected and simply connected.

Let ω be a $\mathfrak{g}$ -valued 1-form on M satisfying the structural equation.

Pick any two points $m_0 \in M$ and $p_0 \in P$.

Then, there is a unique smooth map

$$f : (M, m_0) \longrightarrow (P, p_0), \text{ for which } f^* \omega_P = \omega.$$

This result motivates the idea of Cartan geometries.

From Darboux Contexts to Lie Groups

Theorem

Let $(\mathfrak{g}, P, \omega_P)$ be a Darboux context with P connected, and $\tilde{P} \longrightarrow P$ the universal cover.

Then

- *$\tilde{P}$ has the structure of a Lie group with the Lie algebra $\mathfrak{g}$ and Maurer–Cartan form $\pi^* \omega_P$.*
- *the group of periods $\Phi_{\omega_P}(\pi_1(P, p_0))$ acts by left translations on $\tilde{P}$ as the deck transformation group of π .*

From Darboux Contexts to Lie Groups

Corollary

Let G be a connected Lie group and $\pi : \tilde{G} \longrightarrow G$ its universal cover.

*For any point $\tilde{e} \in \pi^{-1}(e)$, **there exists a unique Lie group structure**, for which $\tilde{e}$ is the identity and π a Lie group homomorphism.*

***Moreover**, $\ker \pi$ is a discrete central subgroup of $\tilde{G}$.*

An Application: a Lie Algebra Integration

Proposition

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let G be connected and simply connected. *Then any homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is induced by a unique homomorphism $\Phi : G \longrightarrow H$.*

Proof.

- Consider an $\mathfrak{h}$ -valued 1-form $\omega := \varphi \circ \omega_G$. It is left-invariant.
- It satisfies the Maurer-Cartan equation.
- So, there is a map $\Phi : (G, e) \longrightarrow (H, e)$, s. t. $\Phi^*(\omega_H) = \omega$.
- Φ is an isomorphism: let f be $L_{\Phi(g_0)^{-1}} \circ \Phi \circ L_{g_0}$, $g_0 \in G$.
- One can show that $f^*\omega_H = \omega$ and $f(e) = \Phi(e)$. Hence, $f = \Phi$.



An Application: a Lie Algebra Integration

Corollary

Let G and H be connected simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and φ an isomorphism of Lie algebras.

Then, φ is induced by a unique isomorphism of Lie groups $\Phi : G \longrightarrow H$.

Proof.

- The existence and uniqueness of Φ follow immediately.
- Since φ is an isomorphism, Φ is a covering map.
- From $\pi_1(H) = \{1\}$, we have Φ being an isomorphism.



An Application: a Lie Algebra Integration

Theorem (Lie's Third Theorem)

Let $\mathfrak{g}$ be a finite dimensional Lie $\mathbb{R}$ -algebra.

Then, there exists a unique (up to isomorphism) connected and simply connected Lie group G with the Lie algebra $\mathfrak{g}$.

Proof.

- By Ado's theorem $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, $\dim V < \infty$.
- Form $H := \langle \exp(x) \mid x \in \mathfrak{g} \rangle \subset G$ via the exponentiation.
- Consider the universal cover $\tilde{H}$



Klein Geometries

Definition

A *Klein geometry* is a pair (G, H) , where G is a Lie group and $H \subset G$ a closed subgroup such that G/H is connected.

Two Klein geometries (G_i, H_i) are *geometrically isomorphic* if there is a Lie group isomorphism

$$\varphi : G_1 \longrightarrow G_2$$

such that $\varphi(H_1) = H_2$.

For example, $(\text{Euc}_n(\mathbb{R}), \text{SO}_n(\mathbb{R}))$, $(\text{SO}_{n+1}(\mathbb{R}), \text{SO}_n(\mathbb{R}))$, and $(\text{L}_{n,1}, \text{SO}_n(\mathbb{R}))$ are three well-known Klein geometries.

Lie-Theoretic Properties of Klein Geometries

With a Klein geometry (G, H) , we have the following list of data:

- ① the smooth manifold $M = G/H$ (a homogeneous space),
- ② the principal H -bundle $H \longrightarrow G \longrightarrow G/H$,
- ③ the Marer-Cartan form $\omega_G : TG \longrightarrow \mathfrak{g}$ satisfying
 - ω_G is a linear isomorphism on each fiber,
 - $R_h^* \omega_G = \text{Ad}(h^{-1}) \omega_G$ for all $h \in H$,
 - $\omega_G(X^\dagger) = X$, where $X^\dagger$ is the fundamental vector field on G .

Infinitesimal Klein Geometries

Definition

This is just a pair $(\mathfrak{g}, \mathfrak{h})$ consisting of a Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$.

The following correspondence

$$\{\text{Klein geometries}\} \longrightarrow \{\text{Klein pairs}\}$$

$$(G, H) \mapsto (\mathfrak{g}, \mathfrak{h})$$

is not surjective.

The Tangent Bundle of Klein Geometries

Let (G, H) be a Klein geometry. Consider a diagram

$$\begin{array}{ccc} T_g(gH) & \xrightarrow[\sim]{\omega_H} & \mathfrak{h} \\ \downarrow & & \downarrow \\ T_g(G) & \xrightarrow[\sim]{\omega_G} & \mathfrak{g} \\ \downarrow & & \downarrow \\ T_{[g]}(G/H) & \xrightarrow{\varphi_g} & \mathfrak{g} / \mathfrak{h} \end{array}$$

The identification map φ_g *depends on the choice* of $g \in G$ over $[g] \in G/H$. This is canonical up to the adjoint action of H on $\mathfrak{g} / \mathfrak{h}$.

The Gauge View of Klein Geometries

Let U be a trivializing neighborhood of $G \longrightarrow G/H$. Then, any two sections called *gauges* are differed by a map $h : U \longrightarrow H$:

$$\sigma_2(u) = \sigma_1(u)h(u) \text{ also written as } \sigma_1 \Rightarrow_h \sigma_2.$$

One can encode a gauge $\sigma : U \longrightarrow G$ via a form $\theta := \sigma^*(\omega_G)$ modulo left multiplication by a fixed element of G . But *for the effective case, this is 1-1*.

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

$$\bar{\theta} : T_u(U) \xrightarrow{\theta} \mathfrak{g} \xrightarrow{\text{pr}} \mathfrak{g} / \mathfrak{h}$$

is an isomorphism.

Model Geometries for Cartan Geometries

A *model geometry for a Cartan geometry* consists of

- an effective infinitesimal Klein geometry $(\mathfrak{g}, \mathfrak{h})$, i. e. the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$ is trivial,
- a Lie group H realizing $\mathfrak{h}$,
- a representation $\text{Ad} : H \longrightarrow \text{GL}(\mathfrak{g})$ extending $\text{Ad}_{\mathfrak{h}} : H \longrightarrow \text{GL}(\mathfrak{h})$.

$\ker(\text{Ad} : H \longrightarrow \text{GL}(\mathfrak{g}))$ is called the *kernel of the model geometry*. If $\ker(\text{Ad})$ is trivial then a model geometry is said to be effective.

Cartan Gauges

Definition

Let $(\mathfrak{g}, \mathfrak{h})$ be a model geometry with group H .

A *Cartan gauge* with this model on a smooth manifold M is a pair

$$(U, \theta_U),$$

$U \subset M$ open, and a 1-form $\theta_U \in \Omega^1(U) \otimes \mathfrak{g}$ satisfying the regularity condition:

$$\bar{\theta} : T_u(U) \xrightarrow{\theta} \mathfrak{g} \xrightarrow{\text{pr}} \mathfrak{g} / \mathfrak{h}$$

is an isomorphism.

Cartan Atlases

Definition

A *Cartan atlas* on a manifold M is a collection $\mathcal{A} = \{(U_\alpha, \theta_\alpha)\}$ of Cartan gauges with a model $(\mathfrak{g}, \mathfrak{h})$ and a group H such that

- *Covering*: the U_α 's form an open cover of M ,
- *Compatibility*: if (U, θ_U) and $(V, \theta_V) \in \mathcal{A}$, then there exists a smooth map

$$k : U \cap V \longrightarrow H$$

such that

$$(\theta_V)_x = \text{Ad}(k(x)^{-1})(\theta_U)_x + (k^*)_x \omega_H$$

for each $x \in U \cap V$.

Two atlases are equivalent if their union is an atlas.

Cartan Geometries: a First Approach

Definition

A *Cartan structure* on a smooth manifold M consists of an equivalence class of Cartan atlases on M .

A *Cartan geometry* is a smooth manifold M together with a specified Cartan structure.

A Cartan geometry is *effective* if the model is effective.

The Curvature of a Cartan Geometry

That's easy to define: for any Cartan gauge (U, θ_U) this is

$$\Theta_U := d\theta_U + \frac{1}{2}[\theta_U, \theta_U].$$

Changing gauges (θ_i, U) , we get the following relation:

$$\theta_1 \Rightarrow_h \theta_2 \text{ implies } \Theta_2 = \text{Ad}(h^{-1})\Theta_1.$$

An easy example: Klein geometries and they are flat.

Other examples: Riemannian geometry, Möbius geometry, projective geometry, etc.

Euclidean and Riemannian Geometries

Definition

Let M be a smooth manifold. An *Euclidean geometry* on M is a Cartan geometry on M modeled on the Euclidean space $(\mathbb{R}^n \rtimes O(n), O(n))$.

A *Riemannian geometry* on M is a torsion-free Euclidean geometry.

$$G \cong \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{R}) \mid A \in O_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$$

$$H \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{R}) \mid A \in O_n(\mathbb{R}) \right\}$$

Euclidean and Riemannian Geometries

$$\mathfrak{g} \cong \left\{ \begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{R}) \mid A + A^t = 0, v \in \mathbb{R}^n \right\}$$

$$\mathfrak{h} \cong \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{R}) \mid A + A^t = 0 \right\}$$

We have an H -invariant splitting $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{p}$ as H -modules, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in \text{Mat}_{n+1}(\mathbb{R}) \mid v \in \mathbb{R}^n \right\}$$

with the Ad -action given by $\text{Ad}(A)v := Av$.

In this setting, the Cartan geometry is said to be *reductive*.

Euclidean and Riemannian Geometries

The splitting induces new splitting:

$$\omega = \omega_{\mathfrak{h}} \oplus \omega_{\mathfrak{p}}, \quad \Omega = \Omega_{\mathfrak{h}} \oplus \Omega_{\mathfrak{p}}$$

Definition

In this set-up, the form $\omega_{\mathfrak{h}}$ is called a *Levi-Civita connection*, and a form $\omega_{\mathfrak{p}}$ a *solder form*.

The structures of H -modules $\mathfrak{h}$ and $\text{Hom}(\Lambda^2(\mathfrak{g} / \mathfrak{h}), \mathfrak{h})$ define all possible types of Riemannian geometries.

From the def'n, we have that a Levi-Civita connection on a principal G -bundle $P \longrightarrow M$ has the following properties:

Euclidean and Riemannian Geometries

From the def'n, we have that a Levi-Civita connection on a principal G -bundle $P \longrightarrow M$ has the following properties:

- ❶ $R_h^* \omega_{\mathfrak{h}} = \text{Ad}(h^{-1}) \circ \omega_{\mathfrak{h}},$
- ❷ $\omega_{\mathfrak{h}}(X^\dagger) = X$ for each $X \in \mathfrak{h}.$

These properties are crucial for the notion of *Ehresmann connections* studied in the Kobayasi–Nomizu book.

Euclidean and Riemannian Geometries

The notion of Riemannian geometries just introduced is closely related with the traditional approach.

Theorem

Let (M, g) be a smooth manifold with a Riemannian metric g .

Then, there is a unique torsion-free Euclidean geometry such that the associated Riemannian metric coincides with g up to a multiplier.

The desired form is $q_p(v) := q(\varphi_p(v))$, where q is a standard on $\mathfrak{g} / \mathfrak{h}$, and $\varphi_p : T_x M \xrightarrow{\sim} \mathfrak{g} / \mathfrak{h}$, $x \in M$, $\pi : P \longrightarrow M$, $p \in \pi^{-1}(x)$.

Other Examples of Cartan Geometries

Möbius geometry, Weyl geometry, projective geometry etc.

One can use methods from differential geometry to solve the PDE's of higher orders: Korteweg–De Vries equations, Kadomtsev–Petviashvili equations etc.

Cartan invented a geometric theory based on projective geometry for solving equations like

$$y'' = A(x, y) + B(x, y)y' + C(x, y)(y')^2 + D(x, y)(y')^3.$$

A Hidden Principal Bundle

Given a manifold M with an effective Cartan structure and a covering family $\mathcal{U} = \{U_i\}$ such that

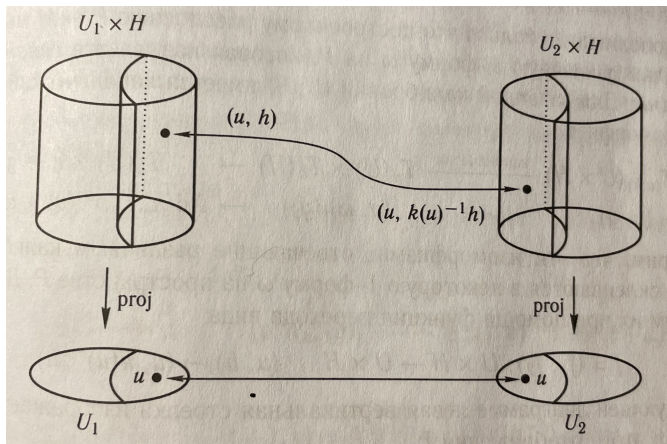
- $U_i \cap U_j$ are connected,
- each of U_i is contained in some open gauge set.

Glue together $\{U_i \times H\}$ for all $U \in \mathcal{U}$ as follows.

If (U_1, θ_1) and (U_2, θ_2) have non-empty intersection there is a *unique gauge equivalence* $\theta_1 \Rightarrow_k \theta_2$ (a *technical lemma needed*).

Then, use the identification map $(u, h) \mapsto (u, k(u)^{-1}h)$ on $(U_1 \cap U_2) \times H$.

A Hidden Principal Bundle



Introducing Cartan Connections

Now, we want to endow the resulting bundle with a Cartan connection.

For a gauge (U, θ) we have a natural isomorphism:

$$T_{(u,h)}(U \times H) \xrightarrow{\sim} T_u U \times \mathfrak{h} \xrightarrow{\sim} \mathfrak{g}.$$

$$(v, y) \longmapsto (v, \omega_H(y)) \longmapsto \text{Ad}(h^{-1}) \circ \theta(v) + \omega_H(y).$$

One can verify that the isomorphisms for all gauges (U, θ) defines a 1-form ω on P .

Properties of Cartan Connections

Proposition

For the Cartan connection ω constructed above the following holds:

- ❶ $\omega_p : T_p(P) \longrightarrow \mathfrak{g}$ is an isomorphism for all $p \in M$,
- ❷ $(R_h)^* \omega = \text{Ad}(h^{-1}) \circ \omega$, for all $h \in H$,
- ❸ $\omega(v^\dagger) = v$ for all $v \in \mathfrak{h}$, where $v^\dagger = (0, V)$ the left invariant vector field on H continued to $U \times H$.

Cartan Geometries: a Bundle Def'n

Definition

A Cartan geometry $\xi = (P, \omega)$ on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with a group H consists of the following data:

- ① a smooth manifold M ,
- ② a principal right H -bundle $P \longrightarrow M$,
- ③ a $\mathfrak{g}$ -valued 1-form ω on P satisfies the following conditions:
 - $\omega_p : T_p P \longrightarrow \mathfrak{g}$ is an isomorphism,
 - $R_h^* \omega = \text{Ad}(h^{-1})\omega$ for all $h \in H$,
 - $\omega(X^\dagger) = X$ for all $X \in \mathfrak{h}$.

The first and second def's are equivalent in the effective case.

Equivalences of Cartan Geometries

Let (P_1, ω_1) and (P_2, ω_2) be Cartan geometries on M_1 and M_2 modeled on a pair $(\mathfrak{g}, \mathfrak{h})$ with a group H . Let

$$f : M_1 \longrightarrow M_2$$

be an immersion covered by an H -bundle

$$\tilde{f} : P_1 \longrightarrow P_2$$

such that

$$\tilde{f}^* \omega_2 = \omega_1.$$

Then f is called a *local isomorphism of geometries*.

If in addition f is a diffeomorphism then it is called an *isomorphism of geometries*.

The Tangent Bundle of a Cartan Geometry

Theorem

Let (P, ω) be a Cartan geometry on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with a group H . Then

$$TM \cong P \times_H \mathfrak{g} / \mathfrak{h}$$

canonically.

Moreover, for each point $p \in P$ with $\pi(p) = x$, there is a canonical linear isomorphism

$$\varphi_p : T_x M \longrightarrow \mathfrak{g} / \mathfrak{h}$$

such that $\varphi_{ph} = \text{Ad}(h^{-1})\varphi_p$.

The Idea of Proof

$$\begin{array}{ccc} T_p(pH) & \xrightarrow[\sim]{\omega_H} & \mathfrak{h} \\ \downarrow & & \downarrow \\ T_p(P) & \xrightarrow[\sim]{\omega} & \mathfrak{g} \\ \downarrow & & \downarrow \\ T_x(M) & \xrightarrow{\varphi_g} & \mathfrak{g} / \mathfrak{h} \end{array}$$

Defining Stuff for Cartan Geometries

One can define such classical notions as tensor fields, covariant differentiations, the Cartan development, etc.

Now, we consider the development for a Cartan geometry (P, ω) .

The Development in Cartan Geometries

Let $\xi = (P, \omega)$ be a Cartan geometry on a manifold M modeled by a Cartan geometry (G, H) .

We want to get *Cartan developments* from M to P .

If ξ is flat, then we are done by the previous considerations.

In the general case, the development of a path depends on not only the homotopy class but also on the path itself.

But along the fibers H , we still have a flat connection!

The Development in Cartan Geometries

Proposition

Let $\xi = (P, \omega)$ be a Cartan geometry on a manifold M modeled on a Klein geometry (G, H) .

Consider a path $\sigma : (I, a, b) \longrightarrow (M, x, y)$. Denote by

$\tilde{\sigma} : (I, a, b) \longrightarrow (P, p, q)$ any lift of σ and $\widehat{\tilde{\sigma}} : (I, a) \longrightarrow (G, e)$ the development of the lifted path in G .

Then,

$$\text{pr}(\widehat{\tilde{\sigma}}) : (I, a) \longrightarrow (G/H, e),$$

where $\text{pr} : G \longrightarrow G/H$, does not depend on the choice of the lift $\widehat{\tilde{\sigma}}$.

The Development in Cartan Geometries

Definition

Let (P, ω) be a Cartan geometry on a manifold M modeled by a Klein geometry (G, H) . Let $\sigma : (I, a, b) \longrightarrow (M, x, y)$ be a path.

The *development* of σ on G/H is defined to be the projection on G/H of the development on G of an arbitrary lift of σ .

Profit: if there is the notion of a straight line in G/H , one can define the notion of geodesic on M as follows.

Definition

A path σ on M is said to be a *geodesic* if its development in G/H is a straight line.

The Development in Cartan Geometries

Via the technique of the Cartan development, one can classify Cartan geometries.

Consider the following example.

An Example: Flat Cartan Geometries

The only obstacle to be locally Kleinian is the curvature.

Theorem

Let M be a flat, effective Cartan geometry modeled with a group H .

Then each point of M has a neighborhood U which is canonically isomorphic as a Cartan geometry to an open subset of the model Klein geometry G/H , where G is any Lie group realizing $\mathfrak{g}$.

Thank you!