

Dynamics and Multivalued Groups

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Introduction

- In different areas of research, multivalued products on spaces appear
- The literature on multivalued groups and their applications is large and includes articles since XIX century mostly in the context of hypergroups
- In 1971, S. P. Novikov and V. M. Buchstaber gave the construction, predicted by characteristic classes. This construction describes a multiplication, with a product of any pair of elements being a non-ordered multiset of n points

Introduction

- It led to the notion of n -valued groups which was given axiomatically and developed by V. M. Buchstaber
- At present, a number of authors are developing n -valued (finite, discrete, topological or algebra geometric) group theory together with applications in various areas of Mathematics and Mathematical Physics

переставить

Buchtaber

- Since 1996, A. P. Veselov and V. M. Buchtaber became develop some applications of n -valued group theory to discrete dynamical systems
- In 2010, V. Dragović showed the associativity equation for 2-valued group explains the Kovalevskaya top integrability mechanism

Introduction

We will talk about

- Symbolic Dynamics
- Tiling theory
- Multivalued Group theory
- their connections and some author's observations

results

Combinatorics on Words Preliminaries

- *Alphabet* A is a finite set, consisting of letters
- A^* stands for the *monoid of finite words* in an alphabet A
- A^ω stands for the set of *right infinite words*
- A word $w \in A^\omega$ is *periodic* if it is of the form $w = uvvv\dots$ for some $u, v \in A^*$
- A word $w \in A^\omega$ is *aperiodic* if it is not periodic
- *Factor* is a finite continuous subword u in $w = \dots u \dots$
- Denote by $|w|$ the length of a word $w \in A^*$

Combinatorics on Words Preliminaries

пусть A и B — алфавиты

- *Morphism* is a map $\mathcal{F} : A^* \rightarrow B^*$ satisfying

$$\mathcal{F}(xy) = \mathcal{F}(x)\mathcal{F}(y)$$

for all words $x, y \in A^*$

- A morphism is defined by the images $\mathcal{F}(a)$ of the letters $a \in A$

Combinatorics on Words Preliminaries

- In some cases, one can define a limit

$$a \rightarrow \mathcal{F}(a) \rightarrow \mathcal{F}(\mathcal{F}(a)) \rightarrow \dots \rightarrow \mathcal{F}^\infty(a)$$

- It is easy to see that the word $w = \mathcal{F}^\infty(a)$ will be *a fixed point*, i. e., $\mathcal{F}(w) = w$

Examples of Morphisms

Example (Fibonacci Morphism)

$$\mathcal{F} : \{0, 1\}^* \rightarrow \{0, 1\}^*, \quad 0 \mapsto 01, \quad 1 \mapsto 0$$

The infinite Fibonacci word $\Phi := \mathcal{F}^\infty(0)$ is

$$\Phi = 0100101001001010010100100101001010010\dots$$

Examples of Morphisms

Example (Thue-Morse Morphism)

$$\mathcal{F} : \{0, 1\}^* \rightarrow \{0, 1\}^*, \quad 0 \mapsto 01, \quad 1 \mapsto 10$$

The Thue-Morse sequence $\mathcal{F}^\infty(0)$ is

$$T = 01101001100101101001011001101001\dots$$

Examples of Morphisms

Example (Tribonacci Morphism)

$$\mathcal{F} : \{a, b, c\}^* \rightarrow \{a, b, c\}^*$$

$$\mathcal{F} : \begin{cases} a \mapsto abc, \\ b \mapsto ac, \\ c \mapsto b \end{cases}$$

The infinite tribonacci word $\mathcal{F}^\infty(a)$ is

abcacbabcbacabcbacabcb...

The Factor Complexity

- *The factor complexity* of a word w is the function $f_w(n)$ defined as the number of its factors of length n
- One can show that ^{for} an infinite word w there exists $C \in \mathbb{N}$ such that

$$f_w(n) \leq C$$

for every $n \in \mathbb{N}$

The Factor Complexity

Theorem (M. Morse and G. Hedlund, 1940)

Let w be an aperiodic infinite word. Then for every $n \in \mathbb{N}$
any

$$f_w(n) \geq n + 1$$

Definition

In the case of equality $f_w(n) = n + 1$, a word w is called
Sturmian

Some easy properties:

- $f_w(n) \leq |A|^n$ where A is an alphabet
- $f_w(n)$ is non-decreasing function

Once Again: The Fibonacci Word

- Consider the Fibonacci word constructed above

$$\Phi = 01001010010010100101001001010010010100...$$

- There is another way to construct Φ
- Consider the following recursive sequence $\{\Phi_k\}$ of *finite Fibonacci words*

$$\Phi_{k+1} = \Phi_k \Phi_{k-1}, \text{ where } \Phi_0 = 0, \Phi_1 = 01$$

- $\{|\Phi_k|\}$ is the *Fibonacci sequence*:

$$|\Phi_k| = F_{k+2}, F_{k+2} = F_{k+1} + F_k, F_0 = 0, F_1 = 1$$

- In this setting $\Phi = \lim_n \Phi_n$

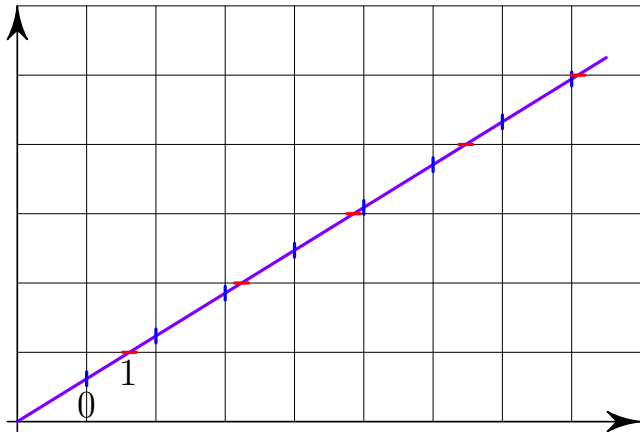
$$\Phi_2 = 010$$

$$\Phi_3 = 01001$$

$$\Phi_4 = 01001010$$

The Fibonacci Word is Sturmian

- It turns out that the Fibonacci word is Sturmian
- It follows from the geometric interpretation of Sturmian words



$y(x) = \psi x, \psi = 1/\varphi$ вставить \phi

$\Phi_5 = 0100101001001$

Some Properties of the Fibonacci Word

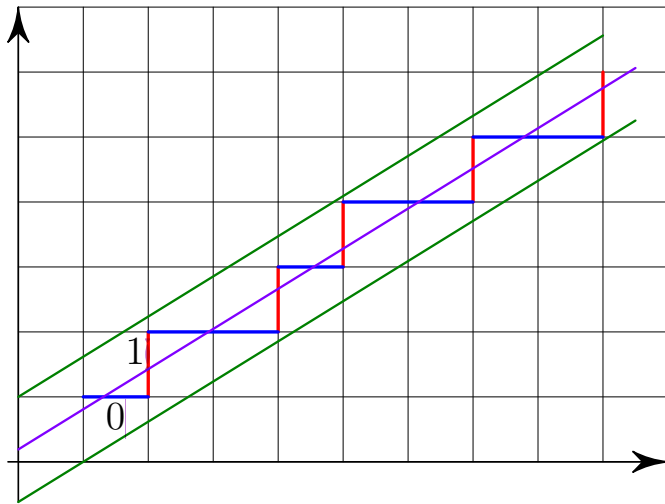
- The factors 11 and 000 are absent in Φ
- The last two letters of a Fibonacci word are alternately 01 and 10
- The n th digit of Φ is

$$2 + \lfloor n\varphi \rfloor - \lfloor (n+1)\varphi \rfloor,$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio

Fibonacci Word and Quasi-Quasicrystals

Cut-and-projection method gives



$$y(x) = \psi x + \frac{1-\psi}{2}, \quad \psi = 1/\varphi$$

определение φ

Balanced Words

Definition

An infinity word w in the alphabet $\{a, b\}$ is called *balanced* if for any two factors u and v of the same length n

$$||u|_a - |v|_a| = 1$$

where $| \cdot |_a$ denotes the number of letters a (the Hamming weight).

- The Fibonacci word is an example of balanced word

$$\phi = 01001010010010100101001001010010010100\dots$$

- For the Thue-Morse word, however, it is not the case: 00 and 11 see, for example

$$T = 01101001100101101001011001101001\dots$$

Geometric Words

Definition

An infinity word in two-letter alphabet is called *geometric* if it encodes intersections of a fixed line $y = \alpha x + \rho$ with vertical and horizontal lines of integer lattice

- If α is rational the dynamics is periodic
- If α is irrational one have a 1-dimensional quasicrystal model
quasi-periodic

Sturmian Words are Geometric

Corollary

For an infinity word in 2-letter alphabet the following conditions are equivalent

- $f_w(n) = n + 1$
- w is aperiodic and balanced

Markov's Result

Theorem (A. A. Markov, 1882)

Let $\alpha = [0; a_1, a_2, \dots]$ be the continued fraction expansion, $\alpha \in (0, 1)$. Then the word $S(\alpha)$ encoded by a line $y = \alpha x$ can be written as follows

$$S(\alpha) = \lim_k S_k(\alpha)$$

where

$$S_k = S_{k-1}^{a_k} S_{k-2}$$

with the initial conditions $S_{-1} = b$ u $S_0 = a$. The letters a and b correspond to vertical and horizontal intersections respectively

For the word length sequence $\{|S_k|\}$ we have $|S_{-1}| = 1$, $|S_0| = 1$ and

$$|S_k| = a_k |S_{k-1}| + |S_{k-2}|$$

Markov's Result

Example

- Consider the line $y = \psi x$ where $\psi = 1/\varphi$, $\varphi = (1 + \sqrt{5})/2$

$$\psi = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

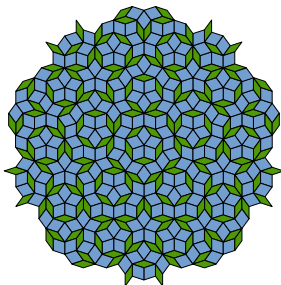
- In this case, $S_n = S_{n-1}S_{n-2}$ — the Fibonacci word

Tilings

Definition

A simple tiling of \mathbb{R}^d :

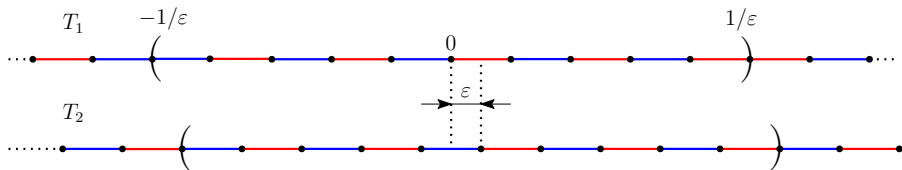
- There are only a finite number of tile types, up to translation
- Each tile is a polytope
- Tiles meet full-edge to full-edge



The ε -closeness

Definition

We say that tilings T_1 and T_2 are ε -close if they agree on a ball of radius $1/\varepsilon$ around the origin, up to translation of size ε or less



Definition

- The *orbit* of a tiling T is the set $\mathcal{O}(T) := \{T - x \mid x \in \mathbb{R}^d\}$ of translates of T
- A *tiling space* Ω is a set that is closed under translation, and complete in the tiling metric
- The *hull* Ω_T of a tiling space T is the closure of $\mathcal{O}(T)$ with respect to the ε -closure property

Tiling Spaces

Example

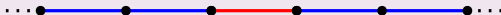
- Consider a simple 1-dimensional tiling T_0 with just one kind of tile. Suppose its length is 1 and its color is blue
- Obviously, $T_0 = T_0 - 1$. So, Ω_{T_0} is a circle



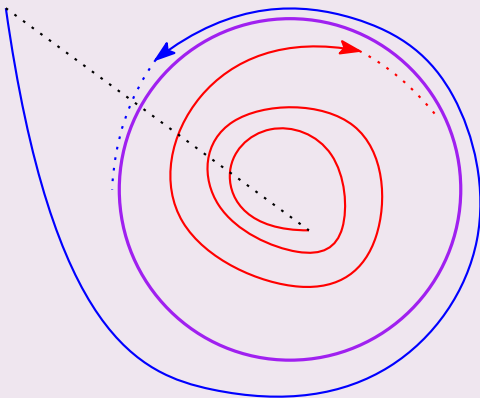
Tiling Spaces

Example

- Consider a 1-dimensional tiling T_1 with one red tile of length 2 and other blue tiles of length 1
- Any tiling with one red tile is in $\mathcal{O}(T_1)$, and hence in Ω_{T_1}
- Tilings with no red tiles are also in Ω_{T_1} by simple reasons
- So, Ω_{T_1} looks like the circle Ω_{T_0} and the line $\mathcal{O}(T_1)$ with both ends of the line asymptotically approaching the circle



Tiling Spaces



Tiling Spaces

Theorem

If T is a simple tiling then Ω_T is compact

- For a tiling T one can approximate the space Ω_T via CW complexes Γ_n from the *Gähler's construction*
- There is a sequence of forgetful maps $f_n : \Gamma_{n+1} \rightarrow \Gamma_n$. The space Γ_n knows about surrounding n layers in some sense
- Hence, one can form an inverse limit and it will be homeomorphic to Ω_T

$$\Omega_T = \varprojlim \Gamma_n$$

- In the case of substitution tilings, it is more convenient to use the *Anderson-Putnam construction* of Γ'_n s

Topological Invariants of Tiling Spaces

- Ω_T has one connected component, but uncountably many path-component
- Each path component in a tiling space is an orbit under \mathbb{R}^d . Such an orbit of an aperiodic tiling is contractible, so $\pi_n(\Omega_T) = 0$ and $H_n(\Omega_n; A) = 0$ for $n > 0$
- Čech cohomology does better

$$\check{H}^* \left(\varprojlim \Gamma_n \right) = \varinjlim \check{H}^*(\Gamma_n) = \varinjlim H^*(\Gamma_n)$$

Example

\check{H}^1 of the Fibonacci tiling space is $\mathbb{Z} \oplus \varphi \mathbb{Z}$, $\varphi = (1 + \sqrt{5})/2$

Symmetric Powers of a Space

- For a topological space X , let $(X)^n$ denote its n -fold symmetric power, i.e., $(X)^n = X^n / \Sigma_n$ where the symmetric group Σ_n acts by permuting the coordinates
- An element of $(X)^n$ is called an n -subset of X or just an n -set. It is a subset with multiplicities of total cardinality n

Example

The spaces $(\mathbb{C})^n = \mathbb{C}^n / \Sigma_n$ and \mathbb{C}^n are identified using the map $\mathcal{S} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ whose components are given by

$$(z_1, z_2, \dots, z_n) \rightarrow \sigma_r(z_1, z_2, \dots, z_n), \quad 1 \leq r \leq n,$$

where σ_r is the r -th elementary symmetric polynomial

- The projectivisation of the map \mathcal{S} induces a homeomorphism between $(\mathbb{CP}^1)^n$ and \mathbb{CP}^n

n -valued Group Structure

An n -valued multiplication on X is a map

$$\mu : X \times X \rightarrow (X)^n : \mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k$$

- *Associativity.* The n^2 -sets

$$\begin{aligned} &[x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n], \\ &[(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z] \end{aligned}$$

are equal for all $x, y, z \in X$

- *Unit.* $e \in X$ such that $e * x = x * e = [x, x, \dots, x]$ for all $x \in X$
- *Inverse.* A map $\text{inv} : X \rightarrow X$ such that

$$e \in \text{inv}(x) * x \text{ and } e \in x * \text{inv}(x) \text{ for all } x \in X$$

The map μ defines an n -valued group structure on X if it is associative, has a unit and an inverse.

Example: 2-valued Group Structure on \mathbb{Z}_+

- Consider the semigroup of nonnegative integers \mathbb{Z}_+
- Define the multiplication $\mu: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow (\mathbb{Z}_+)^2$ by the formula $x * y = [x + y, |x - y|]$
- *The unit:* $e = 0$
- *The inverse:* $\text{inv}(x) = x$.
- *The associativity:* one has to verify that the 4-subsets of \mathbb{Z}_+

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

are equal for all nonnegative integers x, y, z

Homomorphisms of n -valued Groups

Definition

A map $f : X \rightarrow Y$ is called *a homomorphism of n -valued groups* if

- $f(e_X) = e_Y$
- $f(\text{inv}_X(x)) = \text{inv}_Y(f(x))$ for all $x \in X$
- $\mu_Y(f(x), f(y)) = (f)^n \mu_X(x, y)$ for all $x, y \in X$

So, the class of all n -valued groups forms a category

MultValGrp

Reducible n -valued Groups

- For each $m \in \mathbb{N}$, an n -valued group on X , with some multiplication μ , can be regarded as an mn -valued group by using as the multiplication the composition

$$X \times X \xrightarrow{\mu} (X)^n \xrightarrow{(D)^m} (X)^{mn}, \text{ where } D \text{ is diagonal}$$

Definition

An n -valued group on X is called *reducible* if there is an isomorphism $f : X \rightarrow Y$ where Y is an n -valued group with a multiplication $\mu_n = \mu_k^m$, $n = mk$

Homomorphisms of n -valued Groups

Lemma

Let $f : X \rightarrow Y$ be a homomorphism of n -valued groups. Then

- $\ker(f) = \{x \in X \mid f(x) = e_Y\}$ *is an n -valued group*
- $f(x_1) = f(x_2) \Leftrightarrow (f)^n(zx_1) = (f)^n(zx_2)$ *for all $z \in \ker(f)$*
- *Suppose that the map $\text{inv} : X \rightarrow X$ is uniquely defined. Then $\ker(f) = \{e\}$ if and only if any equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$*
- $\text{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$ *is an n -valued group*

Coset Groups

- Let G be a (1-valued) group with the multiplication μ_0 , the unit e_G , and $\text{inv}_G(u) = u^{-1}$
- Let $A \hookrightarrow \text{Aut}G$ be a finite group of order n
- Denote by X the quotient space G/A of G , and denote by $\pi : G \rightarrow X$ the quotient map
- Define the n -valued multiplication $\mu : X \times X \rightarrow (X)^n$ by the formula

$$\mu(x, y) = [\pi(\mu_0(u, v^a)) \mid a \in A]$$

where $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$ and v^a is the image of the action of $a \in A$ on $v \in G$

Theorem

The multiplication μ defines some n -valued coset group structure (G, A) with the unit $e_X = \pi(e_G)$ and the non-ambiguity defined map $\text{inv}(u) = \pi(u^{-1})$ where $\pi(u) = x$

Examples of Coset Groups

- 1 The 2-valued group $(\mathbb{Z}, \mathbb{Z}_2, \varphi)$ on \mathbb{Z}_+
- 2 The additive n -valued group $(\mathbb{C}, \mathbb{Z}_n, \varphi)$ on \mathbb{C}
- 3 Consider $G = \{a, b \mid a^2 = b^2 = e\}$
The interchange of a and b is an element of order 2 of $\text{Aut}G$
Then we have on the set $X = G/A = \{u_{2n}, u_{2n+1}\}, n \geq 0$
where

$$u_{2n} = [(ab)^n, (ba)^n], \quad u_{2n+1} = \{a(ba)^n, b(ab)^n\}$$

the structure of 2-valued group with the multiplication

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}].$$

Thus, X is isomorphic to the 2-valued group on \mathbb{Z}_+ from the point the 1st example

n -valued Dynamics

Definition

n -valued dynamics T on a space Y is a map $T : Y \rightarrow (Y)^n$

- If Y is a state space then the n -valued dynamics T defines possible states $T(y) = [y_1, \dots, y_n]$ at the moment $(t + 1)$ as a state function of y at the moment t

Example

- 1 Consider
 $F(x, y) = b_0(x)y^n + b_1(x)y^{n-1} + \dots + b_n(x), \quad x, y \in \mathbb{C}.$
- 2 The equation $F(x, y) = 0$ defines an n -valued dynamics

$$T : \mathbb{C} \rightarrow (\mathbb{C})^n : T(y) = [y_1, \dots, y_n]$$

$x \text{ and } b_0(x) \neq 0$

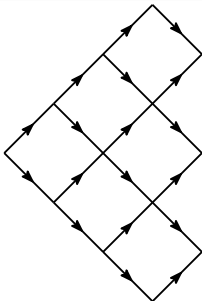
where $[y_1, \dots, y_n]$ — n -set of roots of $F(x, y) = 0$

n -valued Growth Function

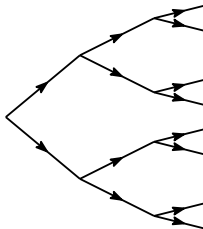
- Let $T: Y \rightarrow (Y)^n$ be an n -valued dynamics. For any $y \in Y$ define the n -valued growth function $\xi_y: \mathbb{N} \rightarrow \mathbb{N}$ where $\xi_y(k)$ — the number of different points in the set $T^k(y)$

Problem

Characterize such n -valued dynamics T that functions $\xi_y(k)$ have polynomial growth for any $y \in Y$



polynomial growth



exponential growth

n -valued Actions

An action of n -valued group X on a space Y is defined by the map

$$\varphi: X \times Y \rightarrow (Y)^n : \varphi(x, y) = x \cdot y = [y_1, \dots, y_n]$$

such that

- for any $x_1, x_2 \in X$ and $y \in Y$ the following n^2 -sets coincide:

$$x_1 \cdot (x_2 \cdot y) = [x_1 \cdot y_1, \dots, x_1 \cdot y_n] \text{ and } (x_1 x_2) \cdot y = [z_1 \cdot y, \dots, z_n \cdot y]$$

$$\text{where } x_2 \cdot y = [y_1, \dots, y_n] \text{ и } x_1 x_2 = [z_1, \dots, z_n]$$

- $e \cdot y = [y, \dots, y]$ for any $y \in Y$

n -valued Cyclic Dynamics

Definition

An n -valued group $X := \langle x \rangle$ is called *cyclic* if it is generated by the only element $x \in X$

Definition

Consider n -valued dynamics $T: Y \rightarrow (Y)^n$ with $X = \langle a \rangle$. The generator a is called *the generator of the cyclic dynamics T*

Theorem (A. A. Gaifullin, P. V. Yagodovskii, 2007)

An n -valued dynamics T has a generator $a \in X$ if and only if there exists such a dynamics $T^{-1}: Y \rightarrow (Y)^n$ that for any $y_1, y_2 \in Y$ the multiplicity of y_2 in $T(y_1)$ equals the multiplicity of y_1 in $T^{-1}(y_2)$

n -valued Cyclic Group Growth Problem

- Let $X = \langle a \rangle$ be a cyclic n -valued group
- Then there is the left action of X on itself

$$T : X \rightarrow (X)^n, \quad T(x) = a \cdot x$$

- Recall $\xi_a(k)$ is a number of different elements in $T^k(a)$

Notation

Denote by $\mathbb{G}_\varphi(G)$ the n -valued group obtained from the construction above for some ordinary group G and some automorphism group element φ

The Case of $\mathbb{Z}/3 * \mathbb{Z}/3$ with $\mathbb{Z}/2 < \text{Aut}$

Proposition

*For the group $\mathbb{Z}/3 * \mathbb{Z}/3 = \langle a, b \mid a^3 = b^3 = 1 \rangle$ and the automorphism $\varphi : a \mapsto b$ the corresponding 2-valued group $\mathbb{G}_\varphi(\mathbb{Z}/3 * \mathbb{Z}/3)$ has the growth function*

$$\xi_{[a,b]}(k) = F_{k+3} - 1 = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{k+3} - \left(\frac{1 - \sqrt{5}}{2} \right)^{k+3} \right) - 1.$$

In particular, the growth is exponential:

$$\xi_{[a,b]}(k) \sim \frac{\varphi^{k+3}}{\sqrt{5}}$$

where $k \rightarrow \infty$ and $\varphi = (1 + \sqrt{5})/2$.

n -bonacci Sequence

Definition

The n -bonacci sequence $\{F_k^{(n)}\}$ is defined recursively as follows:

$$F_k^{(n)} = F_{k-1}^{(n)} + \dots + F_{k-n}^{(n)},$$

initial conditions are $F_0 = \dots = F_{n-2} = 0$ и $F_{n-1} = 1$.

Example

Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

Tribonacci sequence:

$$0, 0, 1, 1, 2, 4, 7, 13, 24, \dots$$

The Case of $\mathbb{Z}/m * \mathbb{Z}/m$ with $\mathbb{Z}/2 < \text{Aut}$

Proposition

The number S_k of new words, appearing on the step k , equals

$$S_k = F_{k+m-2}^{(m-1)}$$

when $k \geq -(m-2)$.

The Case of $\mathbb{Z}/m * \mathbb{Z}/m$ with $\mathbb{Z}/2 < \text{Aut}$

Proposition (M. K.)

*For the group $\mathbb{Z}/m * \mathbb{Z}/m = \langle a, b \mid a^m = b^m = 1 \rangle$, $m \geq 3$ with the automorphism $\varphi : a \mapsto b$ we have*

$$\xi_{[a,b]}(k) \sim \frac{r^{k+1}}{mr - 2(m-1)}$$

*where $k \rightarrow \infty$ and r is the positive root of the polynomial $\chi(\lambda) = \lambda^n - \lambda^{n-1} - \dots - 1$. In particular, $\mathbb{G}_\varphi(\mathbb{Z}/m * \mathbb{Z}/m)$ has the polynomial growth if and only if $m = 2$*

The Case of $(\mathbb{Z}/2)^{*s}$ with $\mathbb{Z}/s < \text{Aut}$

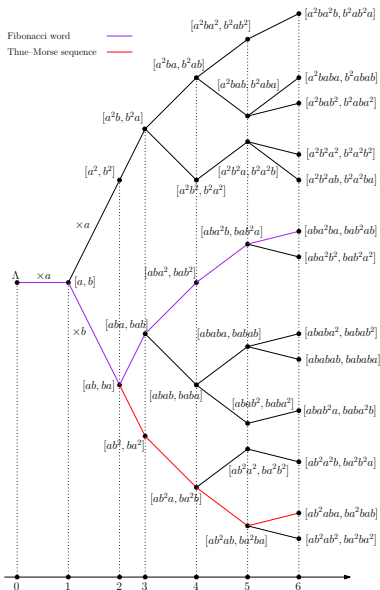
Proposition

*For the group $(\mathbb{Z}/2)^{*s} = \langle a_1, \dots, a_s \mid a_1^2 = \dots = a_s^2 = 1 \rangle$ with the automorphism $a_i \mapsto a_{i+1}$ (indices move modulo s) we have the s -valued group with the growth*

$$\xi_{[a_1, \dots, a_s]}(k) = \begin{cases} \frac{(s-1)^k - 1}{s-2} + 1, & s \geq 3 \\ k + 1, & s = 2 \end{cases}$$

In particular, the growth is polynomial if and only if $s = 2$

$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics



$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics

An algorithm construction of a directed tree Γ , as vertices having the elements of 2-valued group \mathbb{G} :

- 0 We start with the vertex, corresponding to the empty set Λ — the root of our tree
- 1 Add the vertex $[a, b]$ adjacent to the root
- 2 Add two edges to the last vertex: each of them corresponds to an addition a letter (a or b) on the right hand side. Now we have two words of length 2: $[a^2, b^2]$ and $[ab, ba]$

$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics

Definition

We say that a word is *cube-free* (doesn't agree with the common use) if any word in the (natural) normal form of the group $\mathbb{Z}/3 * \mathbb{Z}/3 = \langle a, b \mid a^3 = b^3 = 1 \rangle$

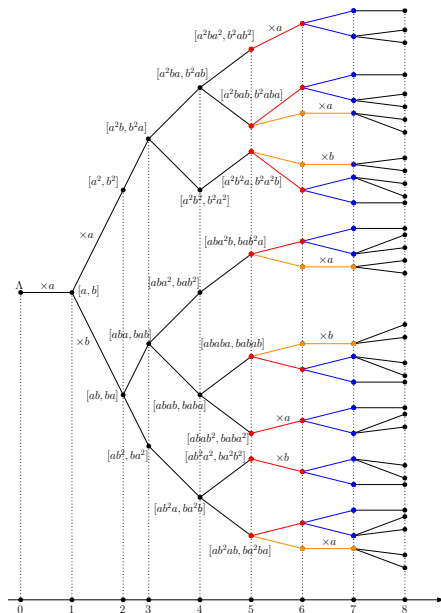
- ④ On the step k we start with all cube-free words of length $k - 1$ and add for each vertex 1 or 2 edges according to the principle:
 - If a word ends with the first power of a letter then we will add 2 edges, corresponding to the multiplications with a and b
 - If a word ends with the square of a letter then we will add exactly one edge, corresponding to the remaining letter
 - The edge, corresponding to the multiplication with a , lies higher than the other one

Properties of Γ

- On the level k of the tree Γ top down, all cube-free words of length k place in lexicographic ascending order and their number is F_{k+1} . Using the binary notation $a \leftrightarrow 0$, $b \leftrightarrow 1$, this order coincides with the natural order on the binary numbers
- If one picks, down to top, the vertex having the number F_k on each k -level of Γ then the resulting vertex sequence will form the route $ab(aab)$ in Γ

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Properties of Γ



Properties of Γ

The latter can be formulated more generally in the following

Proposition (M. K.)

For an infinite cube-free word Ψ , consider the factor sequence $\{\Theta_k\}$ of the form

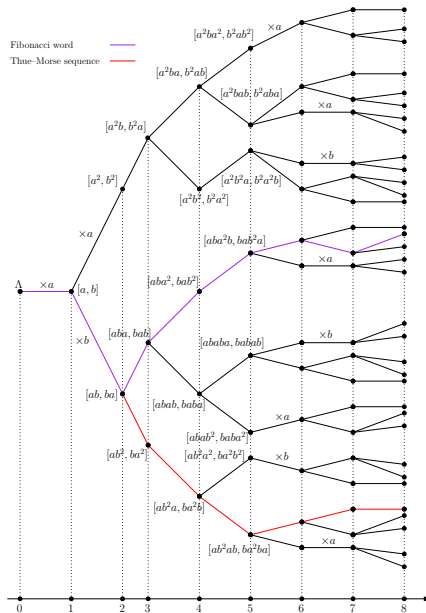
$$\Psi aabaabaab\dots = \Psi(aab)$$

$$\Theta_1 = \Psi, \Theta_2 = \Psi a, \Theta_3 = \Psi aa, \Theta_4 = \Psi aab, \Theta_5 = \Psi aaba, \dots$$

where the last letter of pre-period word Ψ differs from a . Then the number Q_k of cube-free words satisfies the recursive equality, with words being greater or equal Θ_k lexicographically:

$$Q_k = Q_{k-1} + Q_{k-2}.$$

Properties of Γ



Thank you!