n-valued groups and applications

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In various fields of research one encounters
a natural multiplication on a space, say X, under which
the product of a pair of points is a subset of X
(for example, a finite subset).

The literature on multivalued groups and their applications
is very large and includes titles from 19th century,
mainly in the context of hypergroups.

In 1971 S. Novikov and the author introduced a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is an n-multiset, i.e. the unordered set of n points, possibly with repetitions.

This construction leads to the notion of n-valued groups.

Soon afterwards the author gave an axiomatic definition of n-valued groups and obtained the first results on their algebraic structure.
These results have found applications in the well-known problem of algebraic topology.

The condition of n-valuedness is in fact very strong, so, initially it seemed that the supply of interesting examples of n-valued groups is not very rich.

Soon afterwards the author developed the theory of formal,
or local, n-valued Lie groups,
which appeared to be rich of contents and have found important applications in algebraic topology and theory of integrable systems.

In 1990 the author described the structure of an algebraic 2-valued group on the complex projective line $\mathbb{C}P^1$.

Since 1993, Elmer Rees and the author collaborated on the topological and algebraical theory of n-valued groups.

Since 1996 A.Veselov and the author started to work on the applications of the n-valued groups theory to the dynamical systems with discrete time.

At present, a number of authors successfully develope the theory of n-valued groups (finite, discrete, topological, algebraic, algebro-geometric) with applications in various areas of mathematics and mathematical physics.

Symmetric powers of a space.

For a topological space X, let $(X)^n$ denote its n-fold symmetric power, i.e., $(X)^n = X^n/\Sigma_n$ where the symmetric group Σ_n acts by permuting the coordinates.

An element of $(X)^n$ is called an n-subset of X or just an n-set; it is a subset with multiplicities of total cardinality n.

Example. The spaces $(\mathbb{C})^n = \mathbb{C}^n/\Sigma_n$ and $\mathbb{C}^n$ are identified using the map $\mathcal{S} : \mathbb{C}^n \to \mathbb{C}^n$ whose components are given by

$$(z_1, z_2, \ldots, z_n) \to e_r(z_1, z_2, \ldots, z_n),\ 1 \leqslant r \leqslant n,$$

where e_r is the r-th elementary symmetric polynomial.

The projectivisation of the map $\mathcal{S}$ induces a homeomorphism between $(\mathbb{CP}^1)^n$ and $\mathbb{CP}^n$.

n-valued group structure.

An n-valued multiplication on X is a map

$$\mu : X \times X \to (X)^n \ : \ \mu(x,y) = x*y = [z_1, z_2, \dots, z_n], \ z_k = (x*y)_k.$$

Associativity. The n^2-sets:

$$[x*(y*z)_1, x*(y*z)_2, \dots, x*(y*z)_n],$$
$$[(x*y)_1*z, (x*y)_2*z, \dots, (x*y)_n*z]$$

are equal for all $x, y, z \in X$.

Unit. $e \in X$ such that $e*x = x*e = [x, x, \dots, x]$ for all $x \in X$.

Inverse. A map $\mathrm{inv}: X \to X$ such that
$e \in \mathrm{inv}(x)*x$ and $e \in x*\mathrm{inv}(x)$ for all $x \in X$.

The map μ defines an n-valued group structure on X if it is associative, has a unit and an inverse.

First results

Lemma

For each $m \in \mathbb{N}$, an n-valued group on X, with the multiplication μ, can be regarded as an mn-valued group by using as the multiplication the composition

$$X \times X \xrightarrow{\mu} (X)^n \xrightarrow{(D)^m} (X)^{mn}, \quad \text{where } D \text{ is diagonal.}$$

Definition

A map $f\colon X \to Y$ is a homomorphism of n-valued groups if

$$f(e_X) = e_Y,\ f(inv_X(x)) = inv_Y(f(x)) \text{ for all } x \in X,$$

$$\mu_Y(f(x), f(y)) = (f)^n \mu_X(x, y) \text{ for all } x, y \in X.$$

Lemma

Let $f\colon X \to Y$ be a homomorphism of n-valued groups. Then

$$Ker(f) = \{x \in X \mid f(x) = e_Y\} \text{ is an n-valued group.}$$

2-valued group structure on $\mathbb{Z}_+$.

Consider the semigroup of nonnegative integers $\mathbb{Z}_+$.

Define the multiplication $\mu\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2$

by the formula $x * y = [x + y, |x - y|]$.

The unit: $e = 0$.

The inverse: $\mathrm{inv}(x) = x$.

The associativity:

one has to verify that the 4-subsets of $\mathbb{Z}_+$

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

are equal for all nonnegative integers x, y, z.

Additive n-valued group structure on $\mathbb{C}$.

Define the multiplication $\mu\colon \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^n$ by the formula

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n, \quad 1 \leqslant r \leqslant n],$$

where $\varepsilon \in \mathbb{Z}_n$ is a primitive n-th root of unity.

The unit: $e = 0$.

The inverse: $\mathrm{inv}(x) = (-1)^n x$.

The multiplication is described by the polynomial equations

$$p_n(x, y, z) = \prod_{k=1}^{n} \left(z - (x * y)_k\right) = 0.$$

For instance,

$$p_1 = z - x - y, \quad p_2 = (z + x + y)^2 - 4(xy + yz + zx),$$

$$p_3 = (z - x - y)^3 - 27xyz.$$

Coset and double coset groups.

Let G be a (1-valued) group with the multiplication μ_0, the unit e_G, and $\mathrm{inv}_G(u) = u^{-1}$.

Let A be a group with $\# A = n$ and $\varphi : A \to \operatorname{Aut} G$ be a homomorphism to the group of automorphisms of G.

Denote by X the quotient space $G/\varphi(A)$ of G by the action of the group $\mathrm{Im}\varphi$, and denote by $\pi \colon G \to X$ the quotient map.

Define the n-valued multiplication $\mu \colon X \times X \to (X)^n$ by the formula

$$\mu(x, y) = [\pi(\mu_0(u, v^{a_i})), \quad 1 \leqslant i \leqslant n, \quad a_i \in A],$$

where $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$ and v^a is the image of the action of $\varphi(a) \in \operatorname{Aut} G$, $a \in A$ on G.

Theorem

The multiplication μ defines an n-valued group structure on $X = G/\varphi(A)$, called the coset group of (G, A, φ), with the unit $e_X = \pi(e_G)$ and the inverse $\mathrm{inv}_X(x) = \pi(\mathrm{inv}_G(u))$, where $u \in \pi^{-1}(x)$.

In case of $\ker \varphi = 0$ we will identify A with $\varphi(A) \subset \mathrm{Aut}\, G$.

Let $H \subset G$ be a subgroup, and $\#H = n$.

Denote by X the space of double cosets $H \backslash G / H$.

Define the n-valued multiplication $\mu\colon X \times X \to (X)^n$ by the formula

$$\mu(x, y) = \{Hg_1H\} * \{Hg_2H\} = [\{Hg_1hg_2H\} : \ h \in H].$$

Theorem

The multiplication μ defines an n-valued group structure on $X = H\backslash G/H$, called a double coset group of (G, H), with the unit $e_X = \{H\}$ and the inverse $inv_X(x) = \{Hg^{-1}H\}$, where $x = \{HgH\}$.

Each coset group of (G, A, φ), admits a double coset realization on $X = A\backslash G'/A$ as a double coset group of (G', A), where G' is the semidirect product of the groups G and A with respect to the action of A on G by means of φ.

Examples of the coset groups.

(1) The 2-valued group $(\mathbb{Z}, \mathbb{Z}_2, \varphi)$ on $\mathbb{Z}_+$.

(2) The additive n-valued group $(\mathbb{C}, \mathbb{Z}_n, \varphi)$ on $\mathbb{C}$.

(3) Let G be the infinite dihedral group

$$G = \{a, b \mid a^2 = b^2 = e\}.$$

The interchange of a and b generates the automorphism group A, $\#A = 2$. Then

$$X = G/A = \{u_{2n}, u_{2n+1}\}, \quad n \geqslant 0,$$

where $u_{2n} = \{(ab)^n, (ba)^n\}, \quad u_{2n+1} = \{b(ab)^n, a(ba)^n\}$.
Then the multiplication is given by the formula

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}].$$

Thus X is isomorphic to the 2-valued group on $\mathbb{Z}_+$.

(4) Let G be a finite group, $\#G = n$.
Let $A = G$ acts by inner automorphisms

$$g^a = a^{-1} g a, \quad g \in G, \; a \in A.$$

Thus the set of characters of G is an n-valued coset group on $X = G/A$.

Consider $G = \Sigma_3$.

Then $X = \{e, x_1, x_2\}$ is a 6-valued group:

$$\begin{aligned} x_1 * x_1 &= [e, e, e, x_1, x_1, x_1], \\ x_1 * x_2 = x_2 * x_1 &= [x_2, x_2, x_2, x_2, x_2, x_2], \\ x_2 * x_2 &= [e, e, x_1, x_1, x_1, x_1]. \end{aligned}$$

Note, that this 6-valued group on three elements is impossible to reduce to a n-valued group with $n < 6$.

(5) Consider the n-fold direct product G^n of a group G by itself. The group Σ_n acts on G^n by permuting the factors.

Therefore, for any group G, the symmetric product $(G)^n$ is endowed with the structure of an n!-valued coset group.

If G is commutative, with the operation $\mu(g', g'') = g' + g''$, then we have an n!-valued group homomorphism

$$(\mu)^n : (G)^n \longrightarrow G, \quad [g_1, \ldots, g_n] \longrightarrow g_1 + \cdots + g_n,$$

where G is treated as an n!-valued group with the diagonal operation $\mu(g', g'') = [g' + g'', \ldots, g' + g'']$.

In this way we obtain the n!-valued group $\mathrm{Ker}(\mu)^n$.

Take a smooth elliptic curve. It equips the torus T^2 with a commutative group structure.
The construction above produces a structure of an n!-valued group on $(T^2)^n$ for each n.

Thus this construction produces a structure of an n!-valued group on the complex projective space

$$\mathbb{C}P^{n-1} = \mathrm{Ker}((\mu)^n : (T^2)^n \to T^2).$$

For $n = 2$, this yields a structure of a 2-valued group on $\mathbb{C}P^1$.

Using the automorphisms of a smooth elliptic curve, we obtain 2-valued and 3-valued coset group structures on $\mathbb{C}P^1$.

A family of non-coset groups.

Consider the $(2k+1)$-valued group on $X(3) = \{x_0 = e, x_1, x_2\}$ where the multiplication is given by the formulas

$$x_1 * x_1 = [\underbrace{x_1, \dots, x_1}_{k}, \underbrace{x_2, \dots, x_2}_{k+1}],$$

$$x_1 * x_2 = x_2 * x_1 = [e, \underbrace{x_1, \dots, x_1}_{k}, \underbrace{x_2, \dots, x_2}_{k}],$$

$$x_2 * x_2 = [\underbrace{x_1, \dots, x_1}_{k+1}, \underbrace{x_2, \dots, x_2}_{k}].$$

Theorem (S. Evdokimov, 2005)

The $(2k+1)$-valued group $X(3)$ is a coset group if and only if $2k+1 = p^s$, where p is a prime number.

n-valued groups on sets of irreducible unitary representations of groups.

Let G be a finite group, $\#G = m$ and let $\rho_0, \rho_1, \ldots, \rho_k$ be the set of all its irreducible unitary representations, where ρ_0 is the trivial one-dimensional representation.

Consider the decomposition of tensor products of irreducible representations in direct sums of irreducible representations

$$\rho_i \otimes \rho_j = \rho_i \rho_j = \sum_{l=0}^{k} a_{ij}^{l} \rho_l$$

where a_{ij}^{l} is the multiplicity of the representation ρ_l in the product $\rho_i \rho_j$.

We have $a_{ij}^{l} = a_{ji}^{l}$ and, as the classical theory implies,

$$a_{ij}^{0} = \begin{cases} 1, & \text{if } \rho_j = \bar{\rho}_i \\ 0, & \text{otherwise.} \end{cases}$$

Let us denote the dimension of ρ_l by d_l. We have $\sum_{l=0}^{k} d_l^2 = m$.

Let n be the least common multiple (LCM) of $d_i d_j$, $0 \leqslant i \leqslant j \leqslant k$.

Introduce the set of integers $m_{ij}^l = n a_{ij}^l \frac{d_l}{d_i d_j}$. We have

$$\sum_{l=0}^{k} m_{ij}^l = n.$$

Set $x_l = \frac{1}{d_l}\rho_l$ and consider the set $X = \{x_0, \ldots, x_k\}$.

Theorem

The tensor product of representations defines on X a structure of an n-valued group with the product $\mu : X \times X \to (X)^n$, where $\mu(x_i, x_j) = x_i * x_j$ is the n-multiset containing the element x_l with the multiplicity m_{ij}^l, the element x_0 is the unit, and the inverse $\mathrm{inv} : X \to X$ is given by the complex conjugation map, i.e., $\mathrm{inv}(x_l) = \bar{x}_l$, where $\bar{x}_l = \frac{1}{d_l}\bar{\rho}_l$.

Example. $G = \Sigma_3$.

There are irreducible representations $\rho_0, \ \rho_1, \ \rho_2,$
of dimensions $d_0 = 1, \ d_1 = 1$, and $d_2 = 2$, respectively,
with the tensor product table

$$\rho_0\rho_l = \rho_l, \quad l = 0, 1, 2,$$
$$\rho_1^2 = \rho_0, \quad \rho_1\rho_2 = \rho_2, \quad \rho_2^2 = \rho_0 \oplus \rho_1 \oplus \rho_2.$$

So $n = d_2^2 = 4$, and on the set $X = \{x_0 = \rho_0, x_1 = \rho_1, x_2 = \frac{1}{2}\rho_2\}$
we obtain a 4-valued group with the multiplication

$$x_1 * x_1 = [x_0, x_0, x_0, x_0],$$
$$x_1 * x_2 = [x_2, x_2, x_2, x_2],$$
$$x_2 * x_2 = [x_0, x_1, x_2, x_2],$$

the unit $e = x_0$, and the identity map inv.

Note that in this case, the 4-valued group structure
cannot be replaced by a less valued structure.

Generalization of Pontryagin's duality to the case of non-commutative groups.

In the case of a commutative group G, the identification of irreducible unitary representations with the conjugacy classes of G corresponds to the Pontryagin duality.

For a general finite group G, with $\#G = m$, we obtain on these isomorphic sets two structures: the structure of m-valued groups and the structure of n-valued groups where $n = \mathrm{LCM}\{d_i d_j\}$.

For a given noncommutative finite group, this construction yields an n-valued group dual to it.

For example, for $G = \Sigma_3$ we have $m = 6$ and $n = 4$.
Thus, we obtain a 4-valued group dual to the 1-group Σ_3.

Action on a space.

An n-valued group X acts on a space Y if there is a mapping

$$\phi\colon X \times Y \to (Y)^n,$$

also denoted $x \circ y = \phi(x, y)$, such that the two n^2-subsets of Y

$$x_1 \circ (x_2 \circ y) \quad \text{and} \quad (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X$ and $y \in Y$;

and also

$$e \circ y = [y, y, \ldots, y] \quad \text{for all } y \in Y.$$

The coset and double coset actions.

Let G be a certain 1-valued group and $\varphi : \mathrm{A} \to \mathrm{Aut}\,\mathrm{G}$, $\#\mathrm{A} = \mathrm{n}$. Suppose that G and A equivariantly act on some space V, i.e.

$$(\mathrm{g}(\mathrm{v}))^{\mathrm{a}} = \mathrm{g}^{\mathrm{a}}(\mathrm{v}^{\mathrm{a}}), \text{ where } \mathrm{a} \in \mathrm{A},\ \mathrm{g} \in \mathrm{G},\ \mathrm{v} \in \mathrm{V}.$$

There is a natural action of the n-valued coset group $\mathrm{X} = \mathrm{G}/\varphi(\mathrm{A})$ on the space of orbits $\mathrm{Y} = \mathrm{V}/\mathrm{A}$.

Let G be a group and let $\mathrm{H} \subset \mathrm{G}$ be a subgroup, $\#\mathrm{H} = \mathrm{n}$. Suppose G acts on a space V. Then the double coset group $\mathrm{X} = \mathrm{H}\backslash\mathrm{G}/\mathrm{H}$ acts on $\mathrm{Y} = \mathrm{V}/\mathrm{H}$ according to the formula

$$\{\mathrm{HgH}\}\{\mathrm{Hv}\} = [\mathrm{H}(\mathrm{gh})\mathrm{v} : \mathrm{h} \in \mathrm{H}].$$

Algebraic action.

For a given action

$$\phi\colon X \times Y \to (Y)^n,$$

define Γ_x, the graph of the action of $x \in X$, as the subset of $Y \times Y$, which consists of the pairs (y_1, y_2) such that $y_2 \in \phi(x, y_1)$.

Definition

The action of an n-valued group X on an algebraic variety M is called algebraic if the action of any element of X is determined by an algebraic correspondence, i.e., its graph is an algebraic subset in $M \times M$.

It is well known that treating an action of a group G on a space V as a dynamical system leads to fruitful investigations in ergodic theory and geometry.

Among the directions of these investigations, the study of dynamical systems with discrete time, whose evolution is described by a map of V into itself, plays a distinguished role.

It is natural to apply the theory of n-valued groups to n-valued analogues of these systems.

Definition

An n-valued dynamics T with discrete time on a space Y is a continuous map $T : Y \to (Y)^n$.

Thus, if we consider Y as a space of states, then an n-valued dynamics $T : Y \to (Y)^n$ determines possible states $T(y) = [y_1, \dots, y_n]$ at the moment $(t + 1)$ as functions of the state y of the system at the moment t.

Example.

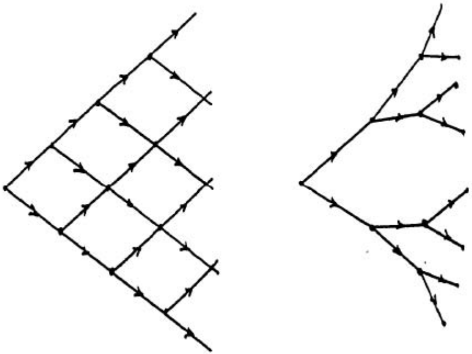
Let $T(x, y) = b_0(x)y^n + b_1(x)y^{n-1} + \cdots + b_n(x)$.

The equation $T(x, y) = 0$ defines an n-valued map (or a n-valued dynamics) $\mathbb{C} \to \mathbb{C}$ under which x is taken to the set of roots $[y_1, y_2, \ldots, y_n]$ of equation $T(x, y) = 0$.

In general case the number of different images of a point grows exponentially with the number of iterations of the map.

In exceptional case the growth is polynomial.

The pictures demonstrates the difference between exceptional and general situations.

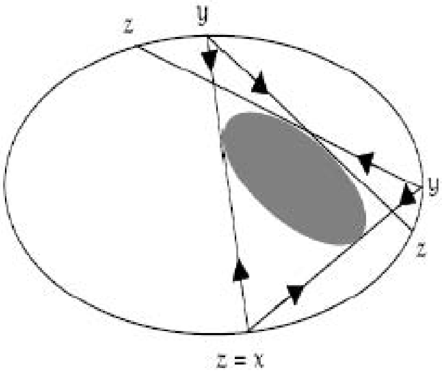


The polynomial

$$T(x,y) = Ax^2y^2 + Bxy(x+y) + C(x^2+y^2) + Dxy + E(x+y) + F,$$

where A, B, C, D, F are constants, defines the 2-valued dynamics, in which the number of different images after the k-th iteration is $k+1$, but not 2^k as one could expect.

The picture explains this fact as the curve $T(x, y) = 0$ describes the geometric situation in the famous Poncelet porism for two conics on the plane.



It is known that for Euler-Chasles correspondence there exists an even elliptic function $f(z)$ of the degree 2, such that if $x = f(z)$ then $[y_1, y_2] = [f(z+a), f(z-a)]$ for some a.

This means that the Euler-Chasles correspondence is the projection of the mapping $z \to z + a$ of the elliptic curve E into itself to the projective line $\mathbb{CP}^1$ which is a coset space $E/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is acting on E as $z \to -z$.

Thus, we have the representation of the two-valued group $\mathbb{Z}_+ = \mathbb{Z}/\mathbb{Z}_2$ with the multiplication

$$x * y = [x + y, |x - y|].$$

Theorem (V.B., A.Veselov, 1996)

All algebraic actions of the two-valued group $\mathbb{Z}_+$ on $\mathbb{CP}^1$ are generated either by the Euler-Chasles correspondence or by a reducible correspondence.

Integrable n-valued dynamics.

Definition

An n-valued dynamics T with discrete time on a space Y is said to be integrable by means of an n-valued group X with a single generator a if the embedding $i : Y \to X \times Y$, $i(y) = (a, y)$, extends to an action φ of the n-valued group X on Y, i.e., the triangle

$$\begin{array}{ccc} Y & \xrightarrow{T} & (Y)^n \\ \downarrow i & \nearrow \varphi & \\ X \times Y & & \end{array}$$

is commutative.

Singly generated n-valued groups.

For 1-valued groups, the only singly generated group are $\mathbb{Z}$ and its quotients, the cyclic groups $\mathbb{Z}_m$.
The variety of singly generated n-valued groups is much wider.

Definition

An n-valued group $(X, *, e, inv)$ is said to be singly generated if there is an element $g \in X$ such that each element $x \in X$ belongs to the multiset $g^k * \bar{g}^l$ for some k and l, where $\bar{g} = inv(g)$.

Let $(X, *, e, inv)$ be a singly generated n-valued group with a generator g such that $\bar{g} \in g^k$ for some $k \geqslant 1$.

Define a graph Γ_g in the following way:
- any $x \in X$ gives a node in Γ_g;
- there is an arrow of weight l from x_1 to x_2 if x_2 belongs to the multiset $g * x_1$ with multiplicity l.

For example, there is an arrow of multiplicity n from e to g.

Finite groups with an irreducible exact representation.

William Burnside (1852–1927)

Theorem (W. Burnside)

Let ρ be an irreducible faithful representation of a finite group G.
Then each irreducible representation of G enters the decomposition of a power $\rho^k = \rho \otimes \cdots \otimes \rho$ into the sum of irreducible summands, for some k.

Corollary

If a finite group G possesses a faithful irreducible representation, then the n-valued group on the set of its irreducible representations is singly generated.

Construction of integrable n-valued dynamical systems with discrete time.

Any simple finite group possesses a faithful irreducible representation.

For example, for $G = \Sigma_3$ the 2-dimensional irreducible representation ρ_2 is faithful, and the 4-valued group on $X = (e, x_1, x_2)$ is singly generated, namely, $x_2 * x_2 = (e, x_1, x_2, x_2)$.

A general condition for the existence of faithful irreducible representations for finite groups see in the paper W.Gaschutz (1954).

Singly generated n-valued groups define integrable n-valued dynamical systems with discrete time.

Associate to a space X the ring $\mathbb{C}[X]$ of all continuous complex valued functions on X. For any positive integer k, we have the canonical map

$$s_k : \mathbb{C}[X] \longrightarrow \mathbb{C}[(X)^k],$$

such that $s_k(f)[x_1, \dots, x_k] = \sum_{i=1}^{k} f(x_i)$.

Definition

Let X be an n-valued group with the multiplication $\mu : X \times X \longrightarrow (X)^n$. The diagonal map

$$\Delta : \mathbb{C}[X] \longrightarrow \mathbb{C}[X \times X] \approx \mathbb{C}[X]\widehat{\otimes}\mathbb{C}[X]$$

is the linear map $\Delta = \frac{1}{n}F$, where

$$F(f)(x, y) = s_n(f)(\mu(x, y)) = \sum_{i=1}^{n} f(z_i)$$

and $\mu(x, y) = x * y = [z_1, \dots, z_n]$.

For $n = 1$ the diagonal map Δ allows one to introduce a Hopf algebra structure on $\mathbb{C}[X]$.

Lemma

The ring of functions $\mathbb{C}[X]$ on an n-valued group X is a coalgebra $(\mathbb{C}[X], \Delta, \varepsilon)$, where Δ is the introduced above diagonal map and the counit $\varepsilon : \mathbb{C}[X] \to \mathbb{C}$ is induced by the unit $e \to X$.

An action $\varphi : X \times Y \to (Y)^n$ of an n-valued group X on Y defines a comodule $(\mathbb{C}[Y], \Delta_Y)$ over the coalgebra $(\mathbb{C}[X], \Delta, \varepsilon)$ where

$$\Delta_Y : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X \times Y] \approx \mathbb{C}[X]\widehat{\otimes}\mathbb{C}[Y],$$

$$\Delta_Y(g)(x, y) = \frac{1}{n}\sum_{i=1}^{n} g(y_i) \;\text{ and }\; [y_1, \dots, y_n] = \varphi(x, y).$$

The $\mathrm{inv} : X \to X$ map axiom implies that there is a map $\mathrm{inv}^{\perp} : X \to (X)^{n-1}$ such that the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{d} & X \times X & \xrightarrow{1\times \mathrm{inv}} & X \times X & \xrightarrow{\mu} & (X)^n \\ & \searrow^{\mathrm{inv}^{\perp}} & & & & \nearrow_{i_n} & \\ & & & (X)^{n-1} & & & \end{array}$$

is commutative; here $i_n[x_1, \dots, x_{n-1}] = [x_1, \dots, x_{n-1}, e]$.

This diagram states that the homomorphism $s_n(\cdot)(\mu(x, \mathrm{inv}\, x)) : \mathbb{C}[X] \to \mathbb{C}[X]$ is split into composition of homomorphisms $(\mathrm{inv}^{\perp})^* \, i_n^* \, s_n(\cdot)$.

If $n = 1$, then this algebraic condition determines the antipode

$$(\mathrm{inv})^* : \mathbb{C}[X] \to \mathbb{C}[X]$$

in the Hopf algebra $\mathbb{C}[X]$.

The multiplication and the comultiplication in a Hopf algebra are related by the fact that the diagonal map is an algebra homomorphism.

In papers by E. Rees and the author, a new algebraic notion, that of an ring n-homomorphism, was introduced and a definition of n-Hopf algebra was given in order to characterize this relation in the case of n-valued groups.

Applications of n-Hopf algebras were developed, based on the following generalization of a classical result about Hopf algebras:

If X is a topological n-valued group, then the ring $\mathbb{C}[X]$ and the cohomology algebra $H^{2*}(X;\mathbb{C})$, in the case $H^{1+2*}(X;\mathbb{C}) = 0$, are n-Hopf algebras.

Corollary

The spaces $\mathbb{C}P^m$ for $m > 1$ do not admit the structure of a 2-valued group.

Frobenius recursion.

Let $f: A \to B$ be a linear map, where A and B are commutative algebras with unit over a field of characteristic 0.

Define, by induction, linear maps

$$\Phi_n(f): A^{\otimes n} \longrightarrow B$$

starting with $\Phi_1(f) = f$

$$\Phi_2(f)(a_1, a_2) = f(a_1)f(a_2) - f(a_1 a_2)$$

and for $n \geqslant 2$

$$\Phi_{n+1}(f)(a_1, \ldots, a_{n+1}) = f(a_1)\Phi_n(f)(a_2, \ldots, a_{n+1}) -$$

$$- \sum_{l=2}^{n+1} \Phi_n(f)(a_2, \ldots, a_1 a_l, \ldots, a_{n+1}).$$

Frobenius n-homomorphisms.

Lemma

If B is a domain and $\Phi_{n+1}(f) \equiv 0$ but $\Phi_n(f) \not\equiv 0$ then $f(1) = n$.

Corollary

If $f : A \to B$ satisfies $\Phi_{n+1}(f) \equiv 0$ and B is a domain then $f(1) \in \{0, 1, 2, \dots, n\}$

Definition (V.Buchstaber, E.Rees, 1997)

A linear map $f : A \to B$ is a Frobenius n-homomorphism if

$$\Phi_{n+1}(f) \equiv 0 \quad \text{and} \quad f(1) = n.$$

Examples.

<u>1-homomorphism</u>

$$f(1) = 1$$

and

$$f(a_1a_2) = f(a_1)f(a_2)$$

that is, a ring homomorphism.

<u>2-homomorphism</u>

$$f(1) = 2$$

and

$$2f(a_1a_2a_3) = f(a_1)f(a_2a_3) + f(a_1a_2)f(a_3) + \\ + f(a_2)f(a_1a_3) - f(a_1)f(a_2)f(a_3).$$

Solution of the recursion.

Let $\mathcal{D}_f(a_1, \ldots, a_{n+1})$ be the determinant of the matrix

$$\begin{pmatrix} f(a_1) & 1 & \ldots & 0 \\ f(a_1 a_2) & f(a_2) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ f(a_1 \cdots a_n) & f(a_2 \cdots a_n) & \ldots & n \\ f(a_1 \cdots a_{n+1}) & f(a_2 \ldots a_{n+1}) & \ldots & f(a_{n+1}) \end{pmatrix}$$

Theorem (V.Buchstaber, E.Rees, 1997)

$$\Phi_{n+1}(f)(a_1, \ldots, a_{n+1}) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \mathcal{D}_f\left(a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}\right).$$

Corollary

The map $\Phi_{n+1}(f)$ is a symmetric multi-linear form.

Multiplicative property.

We denote the sub-algebra of symmetric tensors in $A^{\otimes n}$ by $\mathcal{S}^n A$.

A typical element of $\mathcal{S}^n A$ is

$$a = \sum_{\sigma \in \Sigma_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

and

$$ab = \sum_{\sigma_1, \sigma_2 \in \Sigma_n} a_{\sigma_1(1)} b_{\sigma_2(1)} \otimes \cdots \otimes a_{\sigma_1(n)} b_{\sigma_2(n)}.$$

Theorem (V.Buchstaber, E.Rees)

The linear map $f: A \to B$ is a Frobenius n-homomorphism if and only if

$$\frac{1}{n!} \Phi_n(f) : \mathcal{S}^n A \to B$$

is a ring homomorphism.

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Thank You for the Attention!